# Solutions Manual

Samuel Goldberg (1958)

**Introduction to Difference Equations**

John Wiley

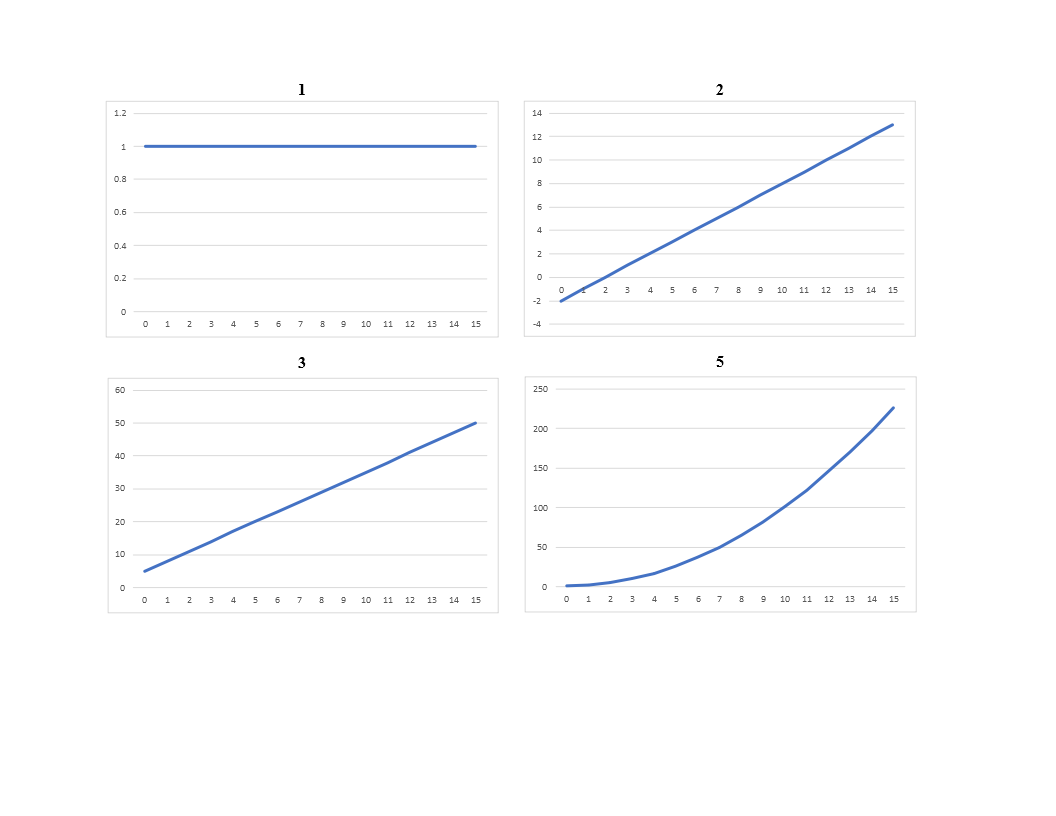
Chapter 1

# 1.1 First difference function (pp. 9 – 18)

In Problems 1.1—1. To 1.1—10, a function is specified by giving its value for any number . Find (a) ; (b) , assuming ; (c) with a difference interval equal to . The symbols and denote constants.

1. . **Answer:**
2. . **Answer:**
3. . **Answer:**
4. . **Answer:**
5. . **Answer:**
6. . **Answer:**
7. . **Answer:**
8. . **Answer:** ;
9. . **Answer:** ;
10. . **Answer:** ;
11. Sketch the graphs of the functions in Problems 1 – 5.

**Answer:** On next page



The function in Problem 4 is equivalent to that in Problems 2 and 3. It is a straight line with .

1. Consider the functions given by and . Show that and . Note the difference in meaning (and value) between and .  
   **Answer:**   
   .  
   .
2. Consider the following table:

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|  | A | A | A | B | B | C | C | C |

Where A, B, and C are numbers. Verify directly from the definition that is a function of . What is the domain of definition of this function?

**Answer:**

Domain – the numbers 0, 1, … ,7

is a function because to every number in its domain there corresponds a single value A, B, or C (see definition of a function on p. 10).

1. Let be the intensity of light required to produce a certain fixed photo-chemical decomposition on an illuminated substance in time . Bartley derives the equation where and are constants. Put and and make a rough sketch of the graph of the function . As increases, not that decreases. Conclude that is negative. Finally, show that your conclusion is not changed if is different from .

**Answer:** On next page

|  |  |
| --- | --- |
|  | Since .  This shows is negative.  If we add the constant to the function, we have  . The in and cancel each other. |

1. Let the function be defined with the domain the set of -values between 0 and 1 inclusive , and suppose that . Show algebraically that and thus conclude that the largest value of is and that this value is assumed when .

**Answer:**

|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | |  |  | | --- | --- | | **x** | **y** | | 0 | 0.0000 | | 0.1 | 0.0900 | | 0.2 | 0.1600 | | 0.3 | 0.2100 | | 0.4 | 0.2400 | | 0.5 | 0.2500 | | 0.6 | 0.2400 | | 0.7 | 0.2100 | | 0.8 | 0.1600 | | 0.9 | 0.0900 | | 1 | 0.0000 | |

1. In a certain population, let denote the *proportion* of the articulate population who openly favor war at time . The proportion who do not openly favor war at time is therefore . Richardson assumes that is proportional to the product , i.e., for some constant . Note that being a proportion, is a number between 0 and 1 inclusive, and use the result of the preceding problem to show that the change in proportion of those in favor of war at time as compared with time is greatest when , i.e., when half the population openly favors war.

**Answer:**

From Problem 15: . The constant has no bearing on the maximum value of reaches its lowest value of 0 when . This proves that the change in proportion of those in favor of war at time as compared with time is greatest when .

1. The demand function in economics specifies the quantity of a certain commodity that people will buy at each price at which it is offered. If denotes price, we let denote the quantity demanded at price . Express the following assumption in suitable mathematical notation: an increase in price causes a fall in the quantity demanded. If the demand function is linear, what does the assumption tell us of the slope of the straight-line graph of this function?

**Answer:** If the function is linear, then and because by the assumption that “an increase in price causes a fall in the quantity demanded,” must be negative.

1. The supply function specifies the quantity of a certain commodity that will be offered for purchase at a given price. Let denote this supply at price . Express in suitable mathematical notation the assumption that an increase in price causes an increase in the quantity supplied. If the supply function is linear, show that the slope of its straight-line graph is positive.

**Answer:** If the function is linear, then . By the assumption that “an increase in price causes an increase in the quantity supplied,” .

1. The supply and demand functions for sugar from 1890 to 1915 were estimated to be given by . If the market price is defined at that value of at which supply and demand are equal, show that the market price is What quantity is demanded at this price? Draw the graphs of the supply and demand functions on the same set of axes and interpret the point of intersection of the resulting straight lines.

**Answer:** Set   
At

1. If is the quantity demanded at the price , the is demanded at the price and denotes the difference in demand due to this price change of amount . The quantity is referred to as the *average* or *arc elasticity* of demand for the price interval to . With the assumption of Problem 17, show that this average elasticity is a negative number. If the demand function is linear, say , show that and conclude that the average elasticity, although dependent upon the initial price , does not depend on the price change, .

**Answer:** Assumption of Problem 17: “An increase in price causes a fall in the quantity demanded.” Hence since the other three terms in the expression are positive.

, which shows that depends on the initial price but not the price change.

# 1.2 Second and higher differences (pp. 18 – 21)

For each of the functions in 1 – 10 of Problems 1.1, find (a) , assuming ; (b) with a difference interval equal to .

1. , **Answer:**  for any , including .
2. . **Answer:**  for any , including .
3. . **Answer:**  for any , including .
4. . **Answer:**  for any , including .
5. **Answer:** . for .
6. . **Answer:** for .
7. **Answer:** for .
8. **Answer:** . for
9. . **Answer:** . for .
10. **Answer:** . for .
11. If and , show that What is your conjecture for for any positive integer ?

**Answer:**

.

.

.

1. Show that

.

.

**Answer:**

.

# 1.3 the operator *e* (PP. 21 – 24)

For each of the functions in 1 – 4 of Problems 1.1 find (a) (b) ; (c) . Use a difference interval equal to .

1. . **Answer:** for all whole numbers 0, 1, …. Therefore, .
2. . **Answer:**

.

1. **Answer:**

.

1. **Answer:**

.

1. If , show that for .

**Answer:** . Assume . Then .

1. Show that (cf. 12 of Problem 1.2)  
      
   .

**Answer:** Use the result from 12 of Problem 1.2 and the definition of from 5 of Problem 1.3. From 12:  
.

.

Now apply the definitions to the result from 12 and obtain the claimed result:

.

1. Show that  
      
   .

**Answer:**

, which proves the first claim.

, which proves the second claim.

1. Let be the function whose value, , at the positive integer is interpreted as the probability of a positive response in the experimental trial. What is the interpretation of ? If , show that . [Cf. assumption (ii) in Example 1 of Chapter, see p. 3: .

**Answer:** ; assume .  
 is the probability of a positive response in the experimental trial.

1. If we define the operator for in the same recursive manner used to define and , i.e., , then show using mathematical induction that .

**Answer:**

By definition, .

# 1.4 Some properties of and

1. Find if is given by
2. **Answer:**
3. **Answer:**
4. **Answer:**
5. Use (1.25, p. 30) to find if
6. **Answer:** Definition 1.6 (p. 29) – . Thus, . Use (1.25, p. 30): . Then, we find   
    for .
7. **Answer:** Use (1.25, p. 30). for .
8. **Answer:** for .
9. Prove the following properties of the operators and :
10. **Answer:**
11. **Answer:**
12. **Answer:**
13. **Answer:** Use property (1.14, p. 22: .  
    For   
    For , see 3(b) above  
    For other
14. **Answer:** Since , this is true for .  
    For .   
    Assume is true for some arbitrary . Can we show that ?  
    . The equality sign in red follows from the assumption that the relationship is valid for and it follows that is must then also be true for .
15. Prove (1.28, p. 31: ) by mathematical induction.  
    **Answer:** Recall that .  
    For For   
    For   
    On page 30, Goldberg derived the result . We see that this formula is satisfied for ,3. Assume it is also satisfied for some and use it repeatedly:
16. **(a)** Write as a sum of factorial functions. (*Hint:* Put . Then put in turn to find   
    **Answer:** **(a)** .  
    Use .  
    Use   
    Use   
    Finally, check that this sum really does add up to   
    **(b)** Find using the result of (a) and check with (1.31, p. 32: ).  
    **Answer:**
17. Show that can be written as the sum of a factorial function for any positive integer . Let  
    . Then show, doing the calculations, that is determined by setting , by setting , etc.

**Answer:** We know that. By choosing the values of the such that the sum , we get the desired result. By setting , the sum is Then setting , the sum is . Then, using the results for and and setting , we find , and so on.

1. Use the result of Problem 6 together with the rules for the difference of factorial functions to prove that .  
   **Answer:** I am not sure how to use Problem 6 to prove the claim. However, proceed as follows,  
   . Then   
   . Using the preceding result, .  
   Assume that . Then, by the first result again,  
      
   . If , we have proved the claim. Otherwise, we continue until we reach . Just to be clear,
2. Prove that if is a polynomial of degree given by (1.21, p. 28: ) then . [*Hint:* Use Theorem 1.3 (p. 28) to show that and then use the result of Problem 7.]  
   **Answer:** Theorem 1.3 states that if , then if . We note that . Of the terms on the RHS, only the last is different from zero. Thus, . We now apply the solution of Problem 7: . Therefore, .
3. Let . Find in the following three different ways and check that each gives the same result: (a) directly from Definition 1.2 (p. 14); (b) by writing as the sum of factorial functions and using (1.25, p. 30); (c) by writing and using (1.30, p.32) and Table 1.1.  
   **Answer:** **(a)**   
      
   **(b)** .

Because this is tedious, I leave the rest.  
**(c)**

1. Prove Theorem 1.4 (p. 32)  
   **Answer:** Theorem 14 states:   
   Proof:
2. Derive the following formula for the difference of the quotient of two functions and :  
    if .  
   **Answer:**
3. Let (income), (investment expenditure), (savings), and (consumer expenditure) be four functions whose values in period are denoted by and , respectively. Assume the two relations and and prove if , and , that and .  
   **Answer:** ; . Since and , we have:
4. Let (government expenditure), (government deficit), and (national income) be functions of time related in such a way that . Make the following inference from this relation: “Then, an extra dollar of deficit would allow a four dollar reduction in tax-financed government spending without affecting the magnitude of national income…”  
   **Answer:** No change in national income means . If deficit increases by one dollar, then .
5. Suppose (price of securities) is related to (money deposits in banks) and (stock securities in banks) in such a way that for each time , a constant . Suppose that in a time interval , taken equal to , increases by an amount and by an amount . Use the result of Problem 11 to show that the . Prove from this that the price of securities rises, falls, or stays the same (in the interval ) according as (the relative increase in money) is greater than, less than, or equal to (the relative increase in securities).  
   **Answer:** We use the result with and . Also note  
   that .  
   . This proves the claim that the price of securities rises, falls, or stays the same (in the interval ) according as is greater, less than, or equal to .
6. Let denote the number of occurrences of a certain response up to the time period , and the number of stimulus elements (of a certain class) which are conditioned to in period . Estes shows that and where , , and are constants. Show that .  
   **Answer:** Note first that . Therefore, .

# 1.5 Equivalence of operators (pp. 34 – 41)

1. If is an operator, prove that .  
   **Answer:** ; this shows that ; this shows that and the two results together prove the claim that .
2. If is a positive integer, prove that . (Cf. 9 of Problems 1.3.)  
   **Answer:** By definition, . The . Assume that for any positive integer . Then . This completes the proof.
3. If in the definition of the factorial function we put , then show that , where and are integers and .  
   **Answer:** , if we put .
4. Prove the following identities for binomial coefficients ( and integers, **a.** . **Answer:**   
   **b.** . **Answer:**   
   **c.** . **Answer:**

.

**d.** . **Answer:** . To prove this, use the binomial theorem:

. Set . This completes the proof (Reference: <http://www.math.ucsd.edu/~gptesler/184a/slides/184a_ch4slides_17-handout.pdf>; Goldberg in a hint, also suggests using the Binomial theorem)

**e.** . **Answer:** Use the Binomial theorem again: . Now set . Thus, . This completes the proof.

1. Show that from a group of individuals, we may select different individuals, different groups of two individuals, different groups of three, etc. [In general, the number of combinations of different objects taken k at a time is . This accounts for the notation , sometimes used in place of the symbol adopted here.]  
   **Answer:** To choose one individual from a group of , we have possible choices. This can be written as . To choose two individuals we have choices for the first and for the second, for a total of . We divide this by two because the choice a,b is the same as b,a. We can write this as . To choose three individuals we have choices for the first, for the second, and for the third for a total of . We divide this by Because the order in which we pick the individuals does not matter. Hence, .  
   To pick individuals we have divided by Because the order in which they are picked does not matter and there a Different ways of picking individuals. We can write this as .
2. Compute and and thus verify Theorem 1.5 (p. 36: ) if
3. . **Answer:** . .
4. . **Answer:**
5. . **Answer:**

In all cases (a, b, and c) yielded the same answer as .

1. Apply (1.43, p. 38: ) to the function for which , a constant, and use identity (e) of Problem 4 to show that (1.43) reduces to the trivial result that if is a positive integer, .  
   **Answer:** The result (e) of Problem 4 shows that this sum is equal to zero. Hence, .
2. Apply (1.43) to the function for which and thus show that . Recognize the indicated sum as the result of applying the binomial theorem to expand so that the equation may be written .  
   **Answer:** Hence, , as claimed in the question. By the binomial theorem, . Therefore, the claim that is true.
3. Apply (1.43) to the function for which a positive integer, and thus derive the formula .  
   **Answer:** The question tells us to use In this problem, .  
   . Therefore, it follows that .
4. The numbers when is put equal to (and ) are denoted by and called *differences of zero*. Using the result of Problem 9, calculate the following differences of zero:

|  |  |  |
| --- | --- | --- |
|  |  |  |
|  |  |  |
|  |  |  |

**Answer:**

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;

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;

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1. Show that for every positive integer . (Hint: Write the result of Problem 9 with and then put .)  
   **Answer:** From Problem 9: .   
   If .
2. In the example following Theorem 1.8 (p. 38) show that . Then check this result by finding the values in the empty cells of the difference table, Table 1.2 (p. 38). (Work to the left from the last column, all of whose entries equal 6.)  
   **Answer:** Use Theorem 1.8: is a positive integer.  
   We are told that and that Using . Using this last result in we find . Since and , it follows that . Finally, using this, we find . Hence, .

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| Table 1.2 | | | | |
|  |  |  |  |  |
|  |  |  |  |  |
|  |  |  |  |  |
|  |  |  |  |  |
|  |  |  |  |  |
|  |  |  |  |  |
|  |  |  |  |  |
|  |  |  |  |  |

# 1.6 Indefinite summation: the operator (pp. 41 –46)

1. Write as the sum of factorials and thus show that .  
   **Answer:** In Problem 1.4, Question 6, we showed that . Hence, . When . Now we calculate  
   . The red equality sign is justified by (1.52, p. 43). Therefore, is correct.
2. Show similarly (cf. 5 of Problems 1.4, p.32) that .  
   **Answer:** From 5, Problem 1.4 . Apply (1.52) and find  
   .
3. Show that, if , **Answer:** In 8 of Problem 1.5 (p. 40) we showed that . Make use of to obtain . Solve this for where .
4. If is a function with period , show that .  
   **Answer:** . Note that . Then:  
   . Since is a function with period . Therefore, . This shows that the claim is correct if is a function with period .
5. If and are each functions with period , prove, directly from the definition of a periodic function, that is also a function with period .  
   **Answer:** By definition of being a periodic function and . Then, , which shows that and is a function with period .
6. Prove that the sum of any finite number of functions of period is a function of period .  
   **Answer:** We can build on the answer to 5 of Problem 1.6. We have shown that the sum of two functions of period is a function of period . Now assume that the sum of functions of period is a function of period . Call this function . By 5 of Problem 1.5, if we add a function of period to , then their sum is also a function of period . Therefore, the sum of any finite number, , of functions of period is also a function of period .
7. Use Theorem 1.9 to show that if , then is a function with period .  
   **Answer:** We us a hint provided by S. Goldberg, that if and is the indefinite sum of some function, then is also the indefinite sum of that function. Now we apply Theorem 1.9 (p. 42) which states that the difference of two functions that are the indefinite sums of the same function, is a function with period . Since , this is the case for .
8. **(a)** Let be the sine function; i.e., the value of for any real number is given by . Using the trigonometric identity with and , show that is a function with period . (Recall that **Answer:** . This proves that is a function with period because for all .  
   **(b)** Let be the cosine function with . Using the identity with and , show that is a function with period .  
   **Answer:** . This proves that is a function with period because .
9. Let be the function given by the equation . Show that but where is any function with period . (This means that the index law of Theorem 1.6, p. 36, cannot be extended to negative exponents.)  
   **Answer:** , where has period . .
10. Find a function for which and are unequal. Compare with Theorem 1.5.  
    **Answer:** Choose . . Thus, . Theorem 1.5 (p. 36) that . Out example demonstrates that this does not extend to negative exponents and does not hold in all cases
11. Use (1.30, p. 32) to prove the following formula for *summation by parts*:  
    .  
    **Answer:** (1.30): .  
    . Therefore:  
    .
12. Using the formula for summation by parts, show that .  
    [Hint: Let , . Then use the result of Problem 3 to find .]  
    **Answer:** From Problem 3: .. But . Therefore: .
13. [Application to finding sums of finite series.] Prove the following theorem:  
    *Let be a given function with domain the set of positive integers and let be any indefinite sum of . Then*   
    [Hint: By hypothesis, for any positive integer . Write this for and add the resulting equations.]  
    **Answer:** , by definition. The Assume that Then, . This completes the proof.
14. By applying the theorem of Problem 13, derive each of the following formulas:  
    **(a)** . [*Hint:* so take **Answer:** From 13 – . Also, from (1.52, p. 43).  
    Then .  
    **(b)** . [*Hint:* ; use the result of Problem 3 to find if .]**Answer:** From Problem 3 – if , . Thus, . Next,   
    .   
    **(c)** . [*Hint*: so choose .]  
    **Answer:** . Thus,   
    . Then, .
15. Write the sum using the summation symbol and thus observe that the result of Problem 14(b) can be stated as follows: the sum of the first terms of a geometric progression with the first term a and common ratio is equal to if . Show that if , the sum equals .  
    **Answer:** Write . If , then . If , then use the result from Problem 14(b) to show that . This completes the proof.

# 2.1 Difference equations – basic definitions (PP. 50 – 55)

1. Write each of the following difference equations in the same form as that of equations (1”) – (5”):  
   **(a)** . **Answer:** Since , it follows that .  
   **(b)** . **Answer:** .  
   **(c)** . **Answer:** .  
   **(d)** . **Answer:**   
   **(e)** . **Answer:** .
2. The difference equations in Problem 1 are all linear with constant coefficients. Find the order of each equation. (Note that your answers do not depends on the set of -values over which the difference equations are defined. This is generally not the case.)  
   **Answer:** (a) 1. (b) 1. (c) 2. (d) 2. (e) 2.

# 2.2 Solutions of a difference equation (pp. 55 – 60)

1. A difference equation of the set and a function are given. In each case, show that the function is a solution of the difference equation. ( and denote arbitrary constants.)  
   **(a)** . **Answer:** is constant. Therefore, .  
   **(b)** . **Answer:** is constant. Therefore, .  
   **(c)** . **Answer:** .  
   **(d)** . **Answer:** .  
   **(e)** . **Answer:** .  
   **(f)** . **Answer:** .  
   **(g)** . **Answer:**   
   **(h)** . **Answer:**   
   **(i)** . **Answer:**   
   **(j)** . **Answer:** .
2. For the difference equations in Problem 1(d), (f), and (j), find the particular solutions for the initial condition . (*Hint:* Use the solution and the initial condition to find the value of for the required particular solution). **Answer:**  
   **(d)** Since .  
   **(f)** Since , .  
   **(j)** Since .
3. For the difference equations in Problem 1(g), (h), and (i), find the particular solutions satisfying the initial conditions and . (Hint: Use the given solution and the initial conditions to obtain two simultaneous equations for the constants and . **Answer:**  
   **(g)** Since and .  
   **(h)** Since and .  
   **(i)** Since and .
4. Consider the difference equation defined over the indicated finite set of -values where is a positive integer greater than . When , the equation involves the value and when , it involves . Therefore, if is a solution of this difference equation, must be defined for . Show that , is a solution of both this difference equation and the boundary conditions .  
   **Answer:** If is a solution to the difference equation, we can write . This shows that is a solution. Now we show that it also satisfies the boundary conditions. .
5. Starting from the solution of Problem 1(c), we may write and . Now eliminate from these two equations to obtain , the difference equation satisfied by the solution with which we started. In this way, starting with the solutions of Problem 1(f) and (j), recover the corresponding difference equations. **Answer:**   
   **(f)** Therefore, .  
   **(j)**   
   .
6. Use the method of Problem 5 to obtain the difference equations of Problem 1(g), (h), and (i). (*Hint:* Use the solutions to write , and and then use these three equations to eliminate the two constants and .). **Answer:**  
   **(g)** . .   
   **(h)** .   
   .  
   **(i)** .   
   .

# 2.3 An existence and uniqueness theorem (pp. 60 –63)

1. The difference equation is linear but not of first order over the indicated set of -values. (Why?) If the initial condition is prescribed, show that is not uniquely determined and there are infinitely many solutions of the difference equation with . Show also that if the value of at any -value different from is prescribed, there is a unique solution of the difference equation.  
   **Answer:**   
   When . This is not a first-order difference equation.  
   If , then can assume any value and the difference equation is satisfied. Each choice of results in a unique solution.  
   If is prescribed, if , , etc., which means, we cannot choose an arbitrary value for if is prescribed because . Thus, if is prescribed, all other values are determined.
2. Consider the linear difference equation of order over the indicated set of three -values. Show that there is no solution of this difference equation for which and Show also that if the prescribed values and are equal (say , there are infinitely many different solutions of the difference equation. Why is Theorem 2.1 (p. 61) not violated by these facts?  
   **Answer:**  
   . But since is arbitrary (with as one possibility), there are infinitely many solutions (not no solutions as the problem states). This does not violate Theorem 2.1 because the theorem requires (in this case ) prescribed values. Here there is only one.  
   . Since can be any number, there are infinitely many solutions.

# 2.4 The equation (pp. 63 –69)

1. Complete the proof of Theorem 2.2 (p. 64) by showing that the function (2.36) is a solution of a difference equation (2.30) when .  
   **Answer:** (2.30) . If then solution is (Theorem 2.2).  
   . If , then . Therefore, if then solution to equation (2.30) is
2. Each of the following difference equations is assumed to be defined over the set of -values . In addition, suppose is prescribed in each case and is equal to . Find the solution of the difference equation, write out the first six value of in sequence form, and describe the apparent behavior of in this sequence.  
   **(a)** . **Answer:** . .  
   **(b)** . **Answer:** . .  
   **(c)** . **Answer:** Write as . . .   
   **(d)** . **Answer:** Write as . .   
   .  
   **(e)** . **Answer:** Write as .   
   .  
   **(f)** **Answer:** Write as . . .
3. Repeat Problem 2, assuming that the prescribed value of is . Again write out the first six values of , starting with . Does the behavior in the sequence change as the initial condition is changed from to ?  
   **(a)** . **Answer:** . is unchanged. The changed only changed the magnitude, not the qualitative behavior of the sequence.  
   **(b)** . **Answer:** . is unchanged, but all values increased by .  
   **(c)** . **Answer:** Write as . . . The changed only changed the magnitude, not the qualitative behavior of the sequence.  
   **(d)** . **Answer:** Write as . .   
   . Still converges to 3 but not from below but from above.  
   **(e)** . **Answer:** Write as . . Still converges to 3.  
   **(f)** **Answer:** Write as . . . The changed only changed the magnitude, not the qualitative behavior of the sequence. Compared to , the magnitude of each value has increased by 3, the factor by which has changed.
4. To prove Theorem 2.3 proceed as follows, starting with the difference equation (2.39): (a) Define the new index by the relation by the relations so that the -values are transformed into -values . (b) Note that where the last equality defines a new function . Show that the difference equation (2.39) can be written in the form (c) Solve the difference equation using Theorem 2.2. [Steps (a) and (b) were designed to reduce the original problem to one we had already solved.] (d) Obtain the results of Theorem 2.3 by replacing by its value .  
   **Answer:** We use the new index and the new function . Then we can rewrite (2.39) -- as . Theorem 2.2 and its Corollary apply to this new difference equation: Replace by and recall that and rewrite the solution for . This proves Theorem 2.3.
5. Solve the difference equation . Find the unique particular solution for which . (Note that if is a solution, then must be defined for .)  
   **Answer:** In the general solution we use Theorem 2.3 with Then: For the particular solution is , .
6. Write the solutions of the difference equations in Problem 2, assuming each to be defined over the set Assume that the prescribed value is Answer: We use our previous results and update them, using Theorem 2.3.  
   **(a)** . **Answer:** .   
   **(b)** . **Answer:** .   
   **(c)** . **Answer:** Write as . .   
   **(d)** . **Answer:** Write as . .   
   **(e)** . **Answer:** Write as .   
   **(f)** **Answer:** Write as . .
7. Often a nonlinear difference equation can be solved by reducing it to a corresponding linear difference equation. For example, consider the equation (2.41) with as a prescribed positive number.  
   **(a)** Show first that if , then for every -value. **Answer:** If is a prescribed positive number, then is defined and positive. If , then is defined and positive for every -value.  
   This allows us to define a new function , the reciprocal of , by the equation (2.42) *.*   
   **(b)** Show that the difference equation (2.41) is transformed into the linear difference equations  
   (2.43) . **Answer:**   
   **(c)** Solve (2.43) and thus show that . Then use (2.42) to find the required solution of the original difference equation: (2.44) ,   
   **Answer:** Solve using 2.36 (when ) . Rewrite this, substituting from (2.42): .
8. Using the same technique as that employed to obtain (2.36), show that the difference equation   
   (2.45) , where and are constants , has the solution (with prescribed)  
   (2.46) , .  
   **Answer:**  
   ; , .   
   If , then . Thus, is a solution.  
   If , then . Thus, is a solution.   
   Proof that   
   . Subtract .  
   . Therefore, .

# Sequences (pp. 69 – 77)

* 1. Consider the following sequences . In each case, write the first five terms of the sequence and determine its type by placing it in one of the classes C1 – C4 or D1 – D4 (Table 2.1., pp. 75-76).  
     **(a)** . **Answer:** (Use Excel to calculate first five terms.) 1, 0.125, 0.037, 0.016, 0.008, 0.005. Type C3 – bounded and monotone decreasing. Converges to 0.  
     **(b)** . **Answer:** . Type D1 –Diverges to .  
     **(c)** . **Answer:** . Type C3 – bounded and monotone decreasing. Converges to 0.  
     **(d)** . **Answer:** . Type D3 – oscillates finitely; bounded.  
     **(e)** . **Answer:** . Type D4 – oscillates infinitely; unbounded.  
     **(f)** . **Answer:** . Type C2 – bounded and monotone increasing. Converges to 1.  
     **(g)** . **Answer:** . Type D2 – diverges to .  
     **(h)** . **Answer:** . Type C4 – damped oscillatory, limit; bounded . Converges to 2.  
     **(i)** . **Answer:** . Type D3 – oscillates finitely; .  
     **(j)** . **Answer:** Type D4 – oscillates infinitely; unbounded.
  2. Determine which of the sequences in Problem 1 are bounded and which unbounded. Is each bounded sequence convergent?  
     **Answer:** First part of question is answered in Problem 1 answers. Second part: No, a bounded sequence is not always convergent, as demonstrated by sequences (d) and (i).
  3. Use Definition 2.5 (p. 71) to prove that the sequence is a null sequence.  
     **Answer:** Let . Choose Then for all .
  4. Use Definition 2.8 (p. 74) to prove that the sequence diverges to .  
     **Answer:** Let be a very large number, say . Thus, . Then for all .
  5. Use Definition 2.6 (p. 72) to prove that the sequence converges and has the limit 1.  
     **Answer:** As the elements 1 in the numerator and 2 in the denominator matter less and less and
  6. Prove that if diverges to and , then diverges to .  
     **Answer:** By definition (2.8, p. 74), if for any arbitrary number there exists such that for all . Then it must be true for for any arbitrary number there exists such that for all .
  7. Consider the pairs of sequences (a) and , (b) and , and thus show if the two sequences, and , both oscillate finitely, then their sum, the sequence , may converge or oscillate finitely. **Answers:**  
     **(a)**   
     **(b)**   
     **(c)** For a sequence to be bounded, there must exist and . Then it is also true that . This shows that the sum will also be bounded. Therefore, if the sum oscillates, it does so finitely. We cannot make a general statement about convergence. However, if both converge, then also converges because .
  8. Consider a pair of sequences (a) and , (b) and , (c) and , (d) and , (e) and , and thus show that if diverges to and diverges to , then may converge, diverge to , diverge to , oscillate finitely, or oscillate infinitely. **Answers:**  
     **(a)** ; the series diverge to and , respectively; their sum is constant.  
     **(b)** ; the series diverge to and , respectively; sum diverges to .  
     **(c)** ; the series diverge to and , respectively; sum diverges to .  
     **(d)** ; the series diverge to and , respectively; sum oscillates finitely.  
     **(e)** ; the series diverge to and , respectively; sum oscillates infinitely.
  9. Consider the pairs of sequences (a) and , (b) and , (c) and , and thus show that if diverges to and oscillates finitely, the product sequence may diverge to , diverge to , or oscillate infinitely. **Answers:**  
     **(a)** diverges to and oscillates finitely. oscillates infinitely. For even and for odd .  
     **(b)** diverges to and oscillates finitely. oscillates infinitely. For even and for odd .  
     **(c)** diverges to and oscillates finitely. oscillates infinitely.
  10. Show that the sequence diverges to but is not monotone increasing.   
      **Answer:** For even . . Thus, for any even . This shows that this sequence is not monotone. To show divergence to , pick any number . Then choose to be the next integer greater than . Then for all .

# Solutions and sequences (pp. 77 – 87)

1. Each of the following difference equations is assumed to be defined over the set of -values 0, 1, 2, …. In each case (i) find the solution of the equation with the indicated value of , (ii) characterize the behavior of the (solution) sequence as in Table 2.2, and (iii) draw a graph of this sequence , carefully labeling both axes and indicating the scale used on each. **Answers:** all graphs are presented at the end  
   **(a)** . (i) . (ii) Constant; row (a) in Table 2.2  
   **(b)** . (i) . (ii) Monotone increasing, diverges to ; row (b) in Table 2.2  
   **(c)** . Rewrite as . (i) . (ii) divergent, oscillates infinitely; row (h) in Table 2.2  
   **(d)** . Rewrite as . (i) . (ii) Monotone decreasing, converges to ; row (d) in Table 2.2.  
   **(e)** . Rewrite as . (i) . (ii) Damped oscillatory, converges to ; row (f) in Table 2.2.  
   **(f)** . (i) . (ii) Monotone decreasing, converges to ; row (k) in Table 2.2.  
   **(g)** . Rewrite as . (i) . (ii) Divergent, oscillates finitely; row (g) in Table 2.2.

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| (a) | (b) | | | (c) |
| (d) | | (e) | | |
| (f ) | | | (g) | |

1. Same as Problem 1 parts (ii) and (iii), for the difference equation in 2 of Problems 2.4, with in each case. **Answers:** Graphs follow at the end.  
   **(a)** . **Answer:** (i) . (ii) Monotone increasing, diverges to ; row (b) in Table 2.2.  
   **(b)** . **Answer: (i)** . (ii) Monotone increasing, diverges to ; row (j) in Table 2.2.  
   **(c)** . **Answer:** Write as . (i) . (ii) Divergent, oscillates finitely; row (g) in Table 2.2.   
   **(d)** . **Answer:** Write as . (i) . (ii) Monotone increasing, converges to 3; row (e) in Table 2.2.  
   **(e)** . **Answer:** Write as . (ii) Damped oscillatory, converges to 3; row (f) in Table 2.2.  
   **(f)** **Answer:** Write as . (i) . (ii) Divergent, oscillates infinitely; row (h) in Table 2.2.

|  |  |
| --- | --- |
| (a) | (b) |
| (c) | (d ) |
| (e) | (f) |

1. Complete the proof of Theorem 2.4 by writing out in detail the proof that diverges to if [*Hint:* Given any positive number , we must find a corresponding number such that for all . Since implies , show that we can take if and any integer greater than otherwise.]  
   **Answer:** Theorem 2.4 (p. 77) concerns the special case when and . We know that in this case the solution is . We use the hint.   
   If , then since for all .   
   If , . If we choose an integer , then for . Note that because and . Also note that if then because .
2. Prove the lemma needed for Theorem 2.5 by mathematical induction. [*Hint:* First show the inequality to be true if . Then, assuming the inequality to be true for , prove it true for ; i.e., assuming (2.64) , prove that . Multiply both sides of (2.64) by the quantity , which is positive (why?), to get . Complete the proof by showing that .]  
   **Answer:** The lemma states that if .  
   If is true. If is true. is true.  
   Assume the inequality to be true for , i.e., . because . Thus, multiplying both sides by does not change the inequality sign, and so:  
   . The final inequality is strictly true for all because then . Thus, we have proved that if then .
3. Prove that if is convergent with limit , then the sequence also converges to and if diverges to or , then does likewise. (Illustration: If is the sequence with general term , then is the sequence with general term ; if is the sequence , then is the sequence . In general, is the sequence with the first term omitted.). **Answer:**   
   Convergence: If is convergent with limit then there exists an integer such that for . Since as well.   
   Divergence: If diverges to or , there exist a number , however large and an integer such that for . Since , as well.
4. If , show that is monotone increasing and always greater than . Then conclude that either converges to some limit (which must be greater than ) or diverges to . To prove that the latter alternative is correct, suppose that actually converges to and deduce a contradiction as follows: use Problem 5 to show that . But it is impossible that since bothand are greater than . Hence diverges to if .  
   **Answer:** Monotone increasing implies . Since this must be true.  
   To prove divergence to we proceed with the proof by contradiction as recommended in the question: suppose that actually converges to and deduce a contradiction as follows: use Problem 5 to show that . But it is impossible that since bothand are greater than . Hence diverges to if .
5. If , show that is monotone decreasing and must therefore converge to some limit , or diverge to . Reject the second possibility by noting that is always positive. Using an argument similar to that of Problem 6, show that and thus prove that .  
   **Answer:** Monotone decreasing implies . Since this must be true.  
   To prove convergence to limit note that for all and therefore cannot diverge to . If there is a limit then . But can only be true if .
6. Complete the proof of Theorem 2.6 by showing that the sequences and are both of the same type as with the exceptions noted. Consider the types C1, … , C4, D1, … , D4 separately. **Answer:**  
   C1 – constant: If   
   C2 – bounded, monotone increasing: . There is an integer and a number such that for all , . If can be used. If and different from such that for all , . Then, for all   
   C3 – bounded, monotone decreasing: . There is an integer and a number such that for all , . If can be used. If and different from such that for all , . Then, for all   
   C2 and C3: it is clear that if .  
   C4 – damped, oscillatory: Essentially the same proof, but if , then   
   D1 – diverges to : There is an integer and a number such that for all ,   
    use if ; use if . Use otherwise. If , then diverges to   
   : If , use . If , then use such that for   
   D2 – diverges to : Basically the same as for D1  
   D3 – oscillates finitely: if , then , if , constant.  
   D3 – oscillates infinitely: if , then , if , constant.

# Simple and compound interest (pp. 87 – 93)

* 1. Find the number of years required for a given sum of money to double itself at (a) simple interest rate 2% per year, (b) compound interest rate 2% per year. [*Hint:* Put in (2.66) and (2.67) and solve for . You will need to know that value of for which . Table of are available for common values of and (or use logarithms and a). From these one finds .] **Answer**  
     **(a)** The formula for simple interest is (2.66): . Using the hint: . Thus, .  
     **(b)** The formula for compound interest is (2.67): . Using the hint: 35.00278878 (cut off at 8 digits).
  2. Show that if the amount is left to accumulate at the interest rate , the resulting sum after periods is . [The quantity is said to be the discounted value of , discounted for compound interest periods.] **Answer:**  
     Use formula (2.67) with . Thus, .
  3. Suppose the constant sum is deposited at the end of each conversion period in a bank which credits interest at the compound rate per period. Let denote the total amount in the account at the end of conversion periods. Show that with . Solve this difference equation and thus show that (2.73) where .  
     [An *ordinary annuity* is a set of periodic payments, usually equal in amount, payable at equal intervals of time, with payments being made at the end of each payment interval. The *amount* (or final values of an ordinary annuity is defined as the sum of all payments, accumulated at compound interest to the time of the last payment. Formula (2.73) gives the amount of an ordinary annuity of payments of at the compound interest rate per period under the assumption that the payment interval equals the conversion period.] **Answer:**  
     **Part 1** – Show that . The first part of the RHS shows the accumulation from earning interest for one period on the assets accumulated during the first periods, and the second part is the ordinary annuity that is added at the end of each period.  
     **Part 2** – Solve Since , the solution is   
     . Recall that .
  4. Using the value and the result of Problem 3, find (a) the amount of an ordinary annuity at 3% after 20 yearly payments of $100, (b) the yearly payment required in order to have 20 payments amount to $5,000 at 3%. **Answer:**  
     **(a)** . In this case.  
     **(b)** 08.
  5. The *term* of an annuity is defined as the time from the beginning of the first payment interval to the end of the last one. The *present value* of an annuity is its value at the beginning of the term. Prove that and thus show that is the present value of an annuity of per period for periods at the interest rate . **Answer:**  
     From (2.77) we know that Therefore, multiply by :  
     . This completes the answer.
  6. Since , then annual payments of $12.95 will amortize a debt of $100 with interest at 5% compounded annually. Construct an amortization schedule for this debt. **Answer:**

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  |  |  | | Annual Payment is Divided into | |
| Year | Outstanding Principal at Start  of the Year | Annual Payment at the End of the Year | | Interest at 5% due at the End of the Year | Repayment of Principal at the End of the Year |
|  | $100.00 | $12.95 | | $5.00 | 7.95 |
|  | 92.05 | 12.95 | | 4.60 | 8.35 |
|  | 83.70 | 12.95 | | 4.19 | 8.77 |
|  | 74.94 | 12.95 | | 3.75 | 9.20 |
|  | 65.74 | 12.95 | | 3.29 | 9.66 |
|  | 56.07 | 12.95 | | 2.80 | 10.15 |
|  | 45.93 | 12.95 | | 2.30 | 10.65 |
|  | 35.27 | 12.95 | | 1.76‬ | 11.19 |
|  | 24.09 | 12.95 | | 1.20 | 11.75 |
|  | 12.34 | 12.95 | | 0.62‬ | 12.33 |
|  | (rounding error) 0.01 |  | |  |  |
| Totals |  | 129.50 | (rounding error 0.01) 29.51 | | 100.00 |

* 1. A debt of $100 to be amortized by equal payments of $30 at the end each year, plus a final partial payment 1 year after the last $30 is paid. If interest is at the rate of 5% compounded annually, construct an amortization schedule and thus show that three full payments are required together with a final partial payment of $22.25 at the end of the fourth year. **Answer:**

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
|  |  |  | | Annual Payment is Divided into | | |
| Year | Outstanding Principal at Start  of the Year | Annual Payment at the End of the Year | | Interest at 5% due at the End of the Year | | Repayment of Principal at the End of the Year |
|  | $100.00 | $30.00 | | $5.00 | | 25.00 |
|  | 75.00 | 30.00 | | 3.75 | | 26.25 |
|  | 48.75 | 30.00 | | 2.44 | | 27.56 |
|  | 21.19 | 22.25 | | 1.06 | | 22.25 |
| Totals |  | 112.25 | 12.25 | | (rounding error) 101.06 | |

* 1. In the *sinking fund* method of repaying a debt, it is assumed that the entire principal remains outstanding until maturity of the debt, that interest on the original principal is paid each period when due, and that the equal periodic payments into the sinking fund accumulate at compound interest to an amount which allows the repayment of the entire principal when it comes due. If is the amount of the debt, the number of payments into the sinking fund, and the compound interest rate earned by the sinking fund, show that or if payments are made as often as interest is converted [*Hint:* Interpret the sinking fund as an annuity whose amount is and use Problem 3.] **Answer:**  
     The periodic (non-interest) payments plus the accumulated compound interest must equal the debt, . Use Problem 3(b): . Since initially there is nothing in the sinking fund (repayment fund), as in Problem 3 and therefore: . This shows that , as claimed.
  2. Prove the following identity: . [Note that this allows the calculation of the sinking fund factor , by an easy subtraction from the corresponding entry in the table for , rather than by a more difficult long division after finding . Note on the Note: This reflects the changes brought about by the computational revolution in the second half of the 20th century.] **Answer:**  
     Reminder: . Therefore, . Subtract the interest rate, : .
  3. A debt of $100 bearing interest at 5 % compounded annually is to be repaid at the end of 10 years by accumulating a sinking fund for this time period. What is the annual sinking fund required if the fund can be invested to yield (a) 5%, (b) 6%? [Note: .] **Answer:**  
     From Problem 8: and from Problem 9: .  
     **(a)** .   
     **(b)** .
  4. Find the annual payment to a depreciation fund, by the sinking fund method, for replacing a machine costing $120 new, having a probable life of 10 years, and a scrap value of $20, if funds can be invested at 5% per annum. [See Problem 10.] If the *book value* at any time is the original cost minus the amount of the depreciation fund at that time, calculate the book value at the end of the first, fifth, and tenth payments. [.] **Answer:**  
     We know that From Problem 10: .   
     Book value, first year:   
     Book value, fifth year:   
     Book value, tenth year:   
     .
  5. Prove: If the fund can be invested at the same rate of interest as that on the debt, the total periodic payment for interest and sinking fund is equal to the periodic payment required to amortize the debt.   
     **Answer:** Intuitively, we can think of the amortization process as borrowing from and then repaying ourselves. This shows that the two approaches are fundamentally the same, and if the interest rates for the two approaches are the same, the regular payment should also be the same.  
     In the sinking fund method, the regular payment to repay a debt over periods is given by  
      (see Problem 8). To this, we add the interest owed on the debt. Therefore, the total payment using the sinking fund method is by the result of Problem 9. The RHS is the period payment for the annuity (equation 2.71, p. 90).
  6. An annuity in which the successive periodic payments increase (decrease) by a constant amount is called an increasing (decreasing) annuity. Let denote the amount accumulated at the end of conversion periods in an increasing annuity whose payments are at the end of conversion periods 1,2,3,… , with interest credited at the compound rate per period. Show that   
      with . This is *not* a difference equation of the type in Theorem 2.7 since is not a constant independent of . But show that   
      is a solution of both the difference equation and initial condition.   
     **Answer:** At the end the first period: ; second period:   
     third period: ;  
     kth period: .  
     . Note that  
     . Substituting this into we have proved the claim . Let since .
  7. Consider the difference equation where denotes the price of bonds in period , and , , and are constants. Solve this equation (assuming specified) and thus show that the sequence converges to if and . Show further that the nature of the approach toward this equilibrium price depends upon the size of , the sequence being monotone if oscillatory if . **Answer:**  
      Use Theorem 2.2 (p.64).   
      and diverges monotonically.  
     . Then, .   
      and monotone convergence.   
      and monotone convergence to .  
      and oscillating and convergence to .  
     .  
      diverges and oscillates.  
      diverges monotonically.

# Economic dynamics, pp. 93 – 98

* 1. Show that if , then and for all . **Answer:**  
     From (2.82) .  
     From (2.83) , . If , then .  
     From (2.77) if . Therefore, .
  2. In the simple multiplier model, two basic assumptions are made: (i) (total income) (investment) (consumption), and (ii) consumption in any period is a linear function of the income of the *preceding* period, or being the marginal propensity to consume (of this year’s consumption with respect to last year’s income). Show that the income function satisfies the difference equation  
      and find the solution of this equation assuming that investment is constant from period to period, say for all . Assuming , show that is convergent with a limiting value . If show that so that “this system follows a path of simple exponential decline of , the difference between and the equilibrium value ” (Boulding, *QJE* 1955: 485-502). **Answer:**  
      follows from and .  
     If , we apply (2.36): . Since and  
     , as claimed.
  3. Following Baumol’s description of the Harrod model, let and denote the income received, total savings, and the entrepreneur’s desired investment, respectively, of a community during period . Translate the following assumptions into mathematical form: (i) the *realized* investment (savings) in any period is a constant proportion, , of the income of that period, (ii) the entrepreneur’s *desired* investment during any period is equal to a constant multiple, , of the increase of the income of that period over the income of the preceding period, and (iii) the investor’s desires are to be realized, i.e., savings in any period are equal to desired investment. From the equations expressing these assumptions, show that if and , then , Solve this difference equation (with prescribed) and then, assuming and show that income will be positive, non-oscillatory, and steadily increasing with time. **Answer:**  
     (i) (ii) , (iii) . For the next step, assume .  
     From (iii) . Thus, as claimed. Solving this difference equation yields:  
     . If , then . In this case, diverges monotonically.
  4. Consider the model of the preceding problem in the case when is greater than . Show that the income sequence is damped oscillatory, oscillates finitely, or oscillates infinitely, according as is greater than, equal to, or less than . **Answer:**  
     From Problem 3: with If , then and oscillates.  
      then and is damped oscillatory.  
      then and oscillates finitely between and .  
      then and oscillates infinitely.
  5. In Problem 3, modify assumption (i) so that the saving during any period are proportional to the income of the *preceding* period. With the other assumptions unchanged, show that now where . Thus prove that if and are both positive, steadily increases as t increases. [Note that the relative magnitudes of s and g are no longer crucial to the behavior of the sequence and compare with Problems 3 and 4.] **Answer:**  
     (i) (ii) , (iii) . For the next step, assume or you get the uninteresting result .  
     From (iii) . Thus, as claimed. Solving this difference equation yields:  
     . If and are both positive for any positive values of and regardless if they are equal or which one is larger. is monotonically increasing.
  6. In Problem 3, modify assumption (i) so that the saving during any period are proportional to the income anticipated in the *next* period and then show, with other assumptions unchanged, that income satisfies the second-order difference equation , . Suppose we have initial prescribed values and . Calculate the first ten terms of the sequence if (a) , and (b) . Verify, in these particular cases, the following general result to be established in the next chapter: if , then diverges to , but if , income oscillates. **Answer:**  
     (i) (ii) , (iii) . From (iii) . Therefore, as claimed.  
     (a)   
     (b)
  7. In Problem 3, modify assumption (ii) to include both additional investment demand proportional to income and investment which is entirely independent of income, i.e., let , where and are constants. Retaining the other assumptions, show that the income now satisfies the difference equation  
     . [Note that this reduces to the difference equation of Problem 3 in the extreme case Solve this equation to find . Verify that if and , then whereas in Problem 3 is constant , now this quantity, “the ratio of the increase of income to the original level of income, must increase as the latter increases.”  
     **Answer:** From Problem 3, (i) . The revised (ii) . Equate (i) and (ii):  
      or . Thus,   
     . To solve this, use (2.36, p. 64): Simplify the -second part of the solution:. Substitute back to get:  
      .   
     To calculate the ratio we cannot use the solution for because it would eliminate and and we could not determine how the ratio changes with income; we need a different approach. If and , (iii) . Divide both sides by : . The first part is constant, and the second part is negative and decreases as income grows. Hence, the ratio increases.
  8. (Expectational Price Cycles [Boulding, Econ Analysis, 1955]) If denotes the amount of money demand for a commodity and the price of the commodity in period , assume a market equation of the form , where is a constant. Suppose further that “people project the trend of prices, so that rising prices lead to the expectation of further rise, and so to an increase in demand, while falling prices lead to an expectation of further fall, and so to a decrease in demand.” To express this in simple form, assume that is equal to some “normal” level plus a factor proportional to the increase in price over the price of the preceding period, i.e., . Derive the difference equation , and find its solution with prescribed. Show that if , then the sequence diverges to (if ) or diverges to (if ), but if then oscillates infinitely, oscillates finitely, or undergoes damped oscillations if is greater than, equal to, or less than ½. In the case of damped oscillations show that the equilibrium (limiting) price is . **Answer:**  
     Given: and . Therefore, .  
      or as claimed. Solve this for using Theorem 2.2 (p. 64): .  
      and diverges. If then and diverges to . If then and diverges to .  
      Damped oscillatory, In this case, .
  9. Suppose that the market price, , is determined by that period’s supply, , according to the relation   
     . At the end of the period , suppliers make an estimate of the next period’s price. Let this expected price in period be and suppose production in period is related to the expected price by the equation .  
     (a) Show that if , then , but that, in general, the actual and expected prices are unequal.  
     (b) Suppose some agency makes a public forecast that the price in period will be and let  
      .  
     [The constant measures the extent to which suppliers have confidence in the public prediction as a better guide to the next period’s price than the current price: means or perfect confidence, means or no confidence.] Show that the value of for which the actual price in period is equal to this public forecast is given by . **Answer:**  
     **(a)** . Thus: . If , then and . Assume , then which makes only if .  
     **(b)**  ; . Since . To answer the question, set . Solve for .

# Inventory analysys (pp. 98 – 103)

* 1. Why is it necessary to write the relation (2.86) with -values starting with rather than with or   
     **Answer:** (2.86) . This becomes if and if . However, and are not defined. is only defined for .
  2. Discuss the behavior of a system with passive inventory adjustments if the equilibrium positions (a) , and (b) , are disturbed by an increase in net investment, , from 100 to 120 units. In each of the two cases, solve the difference equation (2.91), and draw up a table like Table 2.4. Is there a difference in the speed with which the limiting cases are attained in the three cases , and **Answer:**  
     **(a)** . Thus, . Now allow to change to 120: .  
     **(b)** Now allow to change to 120: .  
     Table: Convergence is faster when than when . I will skip the exercise when .

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
| **t** | 1. ) | | | |  | | | |
| **Production** | **Income** | **Sales** | **Inventory** | **Production** | **Income** | **Sales** | **Inventory** |
|  |  |  |  |  |  |  |  |
| 0 | - | 125 | 25 |  | - | 500 | 400 |  |
| 1 | 25 | 145 | 29 | -4 | 400 | 520 | 416 | -16 |
| 2 | 29 | 149 | 29.8 | -0.5 | 416 | 536 | 428.8 | -12.8 |
| 3 | 29.8 | 149.8 | 29.96 | -0.16 | 428.8 | 548.8 | 439.04 | -10.24 |
| 4 | 29.96 | 149.96 | 29.99 | -0.03 | 439.04 | 559.04 | 447.23 | -8.19 |
| 5 | 29.99 | 149.99 | 29.998 | -0.008 | 447.23 | 567.23 | 453.79 | -6.56 |
|  |  |  |  |  |  |  |  |  |
|  | 30 | 150 | 30 | 0 | 480 | 600 | 480 | 0 |

* 1. If denotes the level of inventories at the close of period , show that . With passive inventory adjustments, i.e., , use (2.89, p. 100) and show that . Now use (2.91, p. 100) to eliminate and obtain the equation . Thus, show that , a constant, from which (with ) . **Answer:** . (2.89) . , as claimed. (2.91). ;   
        
     . This can be true only if .  
     However, if: .
  2. With , and , use the result of the preceding problem to show that and   
     , . Calculate from this formula and compare with the values in Table 2.4. **Answer:**  
     . Therefore,   
     . . I used Excel for the calculations. The results agree with those in Table 2.4.
  3. Check the tabular inventory entries made for the two cases of Problem 2 by calculating from the solution found in Problem 3. **Answer:**   
     (a) . . .   
     (b) . . . .

# A probability model for learning (pp. 103 –110)

1. If and , show if and , then . [*Hint:* Establish the inequalities and . **Answer:**  
    reaches its largest possible value if . Hence, is true. It reaches its smallest possible value if Hence,   
    must be true. Therefore, the claim is true.
2. Using (2.98, p. 104) with and , calculate the values of when , and plot the straight-line graph measuring along a horizontal axis and along a vertical axis. **Answer:**  
   . With and , .

|  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | 0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1 |
|  | 0.5 | 0.53 | 0.56 | 0.59 | 0.62 | 0.65 | 0.68 | 0.71 | 0.74 | 0.77 | 0.8 |

1. (a) Solve the difference equation (2.98) in the case , and , and show that when reward and punishment measures are in the ratio 1:4 the limiting probabilities of response and nonresponse are in the same ratio.  
   (b) Generalize this result by showing that when reward and punishment measures are in the ratio the limiting probabilities of response and nonresponse are in the same ratio. **Answer:**  
   **(a)** (2.98, p. 104) . If , and , then  
   . . Thus, .  
   **(b)**   
   . . .
2. With measured along a horizontal and along a vertical axis, plot graphs of the function if (a) (b) and (c) . Assume and show that the larger is, the faster . **Answer:**  
   **(a)**   
   **(b)**   
   **(c)**   
   **Graph:**
3. Repeat the preceding problem with but (a) (b) and (c) .  
   Answer:  
   **(a)**   
   **(b)**   
   **(c)**   
   **Graph:**
4. Suppose that and . If , calculate , show that as , and sketch the resulting *curve of acquisition*. **Answer:**

|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| |  |  | | --- | --- | | n | pn | | 0 | 0 | | 1 | 0.25 | | 2 | 0.375 | | 3 | 0.4375 | | 4 | 0.46875 | | 5 | 0.484375 | | 6 | 0.492188 | | 7 | 0.496094 | | 8 | 0.498047 | | 9 | 0.499023 | | 10 | 0.499512 | | 11 | 0.499756 | | 12 | 0.499878 | | 13 | 0.499939 | | 14 | 0.499969 | | 15 | 0.499985 | | 16 | 0.499992 | | 17 | 0.499996 | | 18 | 0.499998 | | 19 | 0.499999 | | 20 | 0.5 | |  |

1. Consider the difference equation (2.98) with solution (2.105) in the special case . Show that no matter what is prescribed, we have as well as for all . [This is interpreted as insightful or one-trial learning.] How would you interpret the other extreme case in which ? **Answer:**  
   (2.98) . Solution (2.105)   
   : . The value of does not matter because .  
   : . This describes of no learning at all, apparently referred to as one-trial extinction.
2. Let the operators (reward present) and (reward absent) be defined by the equations  
    . Start with and consider the sequence of probabilities obtained by alternately applying the operators and . Show that  
    where . Show further that if , then  
    and thus show that . **Answer:**  
      
    as claimed.  
   : Since and   
   .   
   . Use solution (2.105) and obtain   
    .
3. Generalize the preceding problem to the *fixed-ratio reinforcement* schedule where only every th response is rewarded. That is, starting with , we obtain the sequence . Show that   
    where . Show further that if , then  
    and thus show that . If and are fixed, then what happens to this limiting value as ? **Answer:**  
    as claimed. Then, and . Use solution (2.105) and obtain . and Therefore, .
4. Denote by the probability that a relevant cue has been conditioned at the beginning of the th trial. Show that the hypothesis “on each trial of a given problem a constant portion, , of unconditioned relevant cues become conditioned” may be translated by the equation . Note that this is a difference equation for the function considered as a function of . Solve this equation with the prescribed initial condition and show that . **Answer:**  
      
    since We used Theorem 2.3 (see (2.40), p. 67).
5. Letting denote the number of elements in a population of stimuli, a class of behaviors, the mean number of elements from effective in any one trial, and the expected number of elements from which are conditioned to in trial number , Estes (1950, *Psych Rev*, 94-107) derives the equation  
   . Solve this difference equation (with prescribed) to obtain  
   . **Answer:**  
   . Now solve this difference equation:  
    . This can also be written as which shows the claim to be true.
6. In deriving expressions for spontaneous recovery and regression, Estes (1955, *Psych Rev*, 145-154) introduces the following stimulus fluctuation model. A total set of stimuli, , is subdivided into two sets, an available set and an unavailable set . In each period of time, an element in may either become a member of (with probability ) or remain in (with probability ). Similarly, an element originally in moves into or stays in with probabilities and respectively. Let denote the probability that a given element of is in at the end of period . This element is in at the end of period if and only if one of the following occurs: (a) it was in at the end of period and stayed there; (b) it was in at the end of period and then moved into . Thus show that .  
   Solve this difference equation to find (formal [1] in Estes’ paper) where  
   . **Answer:**  
   . Now use to write . This can be rewritten as .
7. Reinterpret the preceding problem, using the following new meanings for the symbols introduced there. Let be the panel of individuals who ae asked a certain question (capable of being answers “yes” or “no”). Set is divided into two sets: (those individuals answering “yes”) and (those answering “no”). In two successive panel polls in which the same questions is being asked, an individual may move from to or stay in , and an individual in may move to or stay in . That is, each person, whether answering “yes” or “no” in any poll, may keep or change his opinion in the next poll. The probability is the probability that an individual answers “yes” in poll number . Show that if and , , so that as . **Answer:**  
   From Problem 12:   
    and .
8. Show that the difference equation for the function , , where and are constants independent of , has the solution . The parameter is actually a probability so . Show that as unless . **Answer:**  
   .
9. Show that the difference equation with has the solution ( and being constants) where . **Answer:**  
   . Solve this difference equation:  
   . Use . Note that   
   .

# Geometric growth (pp. 110 – 116)

1. For the utility-wealth relation given by the Bernoulli hypothesis (2.107) with solution (2.115), assume , and . Since , we may plot a graph of this utility-wealth relation. Do this, plotting on the vertical and on the horizontal axis. [For example, when we obtain the point (); when we have (), etc. **Answer:**  
   (2.107, p. 111) . If . Solving this difference equation yields (2.115, p. 114) . With , and : . The graph follows.
2. Sketch graphs as in the preceding problem, leaving the values of and unchanged, but with (a) and (b) . Observe as increases, the rate of increase of decreases, the graphs approaching the constant (no-growth) horizontal line . **Answer:** (a) (b)
3. Show that a labeling of the axes in the preceding problems allows the graphs sketched there to be interpreted as graphs of the response-stimulus relations given in the solution (2.116) of the Weber-Fechner law. **Answer:**  
   (2.115, p. 114) . (2.116, p. 114) . Replacing with makes the graphs representations of (2.116), as stated in the question.
4. Suppose the members of a group are to be ranked according to the extent to which each participates in some group task. Let denote the value of some quantitative (actual or estimated) measure of participation for an individual in rank , and let , where and are positive constants and .   
   (a) Show that is the measure of participation of an individual of rank 0.  
   (b) Show that so that is the ratio of participation for any rank to the participation for the next lower rank.  
   (c) Show that the function decreases geometrically. **Answer:**  
   **(a)** If , then .  
   **(b)** If , then .  
   **(c)** If , then . Also: . decreases at a constant rate.
5. Denote by the number of individuals in a certain population at the end of the year . Show that the differeence equation expresses the assumption that the population undergoes a 2% relative increase in size per year, or, equivalently, that the increase in size in any year is proportional to the size at the beginning of the year, the constant of proportionality being 0.02. Solve this difference equation to obtain , where denotes the initial population size. Note that the population grows geometrically. To find the number of years required for doubling the population size, put and solve for . Compare with 1(b) of Problems 2.7. **Answer:**  
   . Solving this difference equation yields:  
   . Use the hint and put : .  
   . This is the same result as 1(b) of Problems 2.7.t
6. Suppose the natural increase in population size of the preceding problem is offset by a constant annual loss of ten individuals due to adverse conditions. Show that and solve this difference equation to find . To find the number of years required for doubling the population size put and solve for . Show that this value of satisfies the equation  
    and thus depends on the initial population size . (Cf. the preceding problem.) Show that this value of is 41 years if . (Note: correct to two decimal places.) **Answer:**  
    as claimed. To find the doubling time:   
   . Assume . Then . Then  
   .
7. Suppose the natural increase in population size of Problem 5 is augmented by an annual immigration of ten individuals. Formulate the appropriate difference equation for and show that  
   . Show that approximately 31 years are required for an initial population of 2500 individuals to double itself. (Note: .) Does the number of years depend upon the initial population size? **Answer:**  
      
   , as claimed.  
   For the doubling of the population: . With , this results in  
   .  
   Does the doubling depend on the initial population? . Therefore, depends on .
8. Suppose a well produces barrels of oil per day during the time interval and let there be a constant percentage relative decline in production assumed to occur only at the end of each time interval. If the numerical value of this fixed percentage equals , show that so that decreases geometrically according to the formula . **Answer:**  
   By assumption we have .  
   From the preceding line: .
9. According to Boulding, the profit-making process may be thought of as the process by which the total net worth of the economy grows. Suppose the net worth at the end of year is denoted by and assume net worth has a constant rate of growth of 5% per year. Show that the function satisfies the difference equation and solve to find . Sketch a graph of the growth of net worth assuming is the base year and . Measure along the horizontal and along the vertical axis. **Answer:**  
   By assumption, .Suppose time   
   Thus, . Graph follows:
10. Suppose time is divided into practice periods of constant length and that the level of attainment in certain learning experiments is measured by , the number of successful acts performed in practice peridod . If is a constant measuring the limit of attainment in terms of successful acts performed per period, then is defined as the margin of attainment in period . If the relative change of the margin of attainment is a constant decrease of magintude , show that and solve the difference equation to obtain . With and , calculate , graph the margin of attainment as a function of the number of practice periods, and show that y decreases geometrically.  
    **Answer:**. By assumption,   
    Therefore: .

|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| |  |  | | --- | --- | | **k** | **yk** | | 0 | 10.000 | | 1 | 8.000 | | 2 | 6.400 | | 3 | 5.120 | | 4 | 4.096 | | 5 | 3.277 | | 6 | 2.621 | | 7 | 2.097 | | 8 | 1.678 | | 9 | 1.342 | | 10 | 1.074 | |  |

# Linear difference equations with constant coefficients

# 3.1 Some basic theorems (pp. 121 –128)

1. In the general forms (3.5) through (3.7) for the linear difference equations with constant coefficients of orders 1, 2, and 3, identify the values of the coefficients and the right-hand member if  
   **(a)** . **Answer**: .  
   **(b)** . **Answer**: .  
   **(c)** . **Answer**:   
   **(d)** . **Answer:**   
   **(e)** **Answer:**   
   **(f)** **Answer:**
2. Write the general form of the linear difference equations with constant coefficients of orders 4 and 5. Assume in each case that the leading coefficient is 1. [*Hint*: Use (3.4) with and.] **Answer:**  
   .  
   .
3. Show that each function is a particular solution of the difference equation in the corresponding part of problem 1:

|  |  |
| --- | --- |
| (a)  (b)  (c) | (d)  (e)  (f) |

**(a)** . **Answer**: .  
**(b)** . **Answer**: .  
**(c)** . **Answer**:   
**(d)** . **Answer:**   
**(e)** **Answer:**   
**(f)** . Answer:

1. Using the results of Problems 3 and Theorem 3.3, find the general solutions of the difference equations in Problem 1(a), (b), and (c). In each case, also find the particular solution for which . **Answer:**  
   **(a)** . **Answer**: . General solution: .   
   **(b)** . **Answer**: . General solution: .  
   **(c)** . **Answer**: . General solution:
2. Carry out the proof of Theorem 3.1 (p. 122) in the two cases and , i.e., for the first- and third-order homogenous equations corresponding to the complete equations (3.5) and (3.7). **Answer:**  
   The homogenous version of (3.5) is . If and are solutions, then . Therefore, the linear combination , as well.  
   The homogenous version of (3.7) is . If and are solutions, then . Therefore, the linear combination is also a solution.
3. Prove the corollary to Theorem 3.1 (p. 123) by mathematical induction, writing out the details in the case of the difference equation or order 2. **Answer:**  
   . Assume: and are solutions. Then (Problem 5), is also a solution. If is also a solution, then is also a solution by Theorem 3.1. Now assume that are solutions and is also a solution. Then if is a solution, we have already proved that is also a solution. This complete the proof by induction.
4. In the hypothesis of Theorem 3.3, it is assumed that the difference equation is of order 1, i.e., the coefficient . Where is this fact used in the proof? **Answer:**  
   On page 125, we state . If and is indeterminate.

# 3.2 Fundamental sets of solutions (pp. 128 – 134)

1. Show that the functions and given by and are solutions of the difference equation and that they form a fundamental set. Find the general solution of the difference equation and a particular solution for which and . **Answer:**  
   . Thus both are solutions.  
    and   
   the two solutions form a fundamental set  
   General solution: ; and and .   
   Particular solution:
2. Find the particular solution of the difference equation in Problem 1 which satisfies the initial conditions and . Note that this is the same solution as that obtained in Problem 1. Explain. **Answer:**  
   ; and and . The reason is that and for .
3. Use the result of 3(d) of Problems 3.1 and Problem 1 to find the general solution for the difference equation   
   . Find a particular solution of this equation for which and . **Answer:**  
   From Problems 3.1, 3(d) , and from 1. and . Therefore, the general solution is . To find the particular solution:  
   ;   
   Particular solution:
4. In each of the following parts, a difference equation and three functions , and , are given. Show in each case that (a) and are solutions of the corresponding homogenous equation; (b) and form a fundamental set of solutions; (c) is a solution of the complete equation. The (d) use Theorem 3.6 to write the general solution of the complete equation, and (e) find the particular solution satisfying the initial conditions and .  
   **(i)** **Answer:** (a)   
   (b) ; we have a fundamental set of solutions.  
   (c) . is a solution of the complete equation.  
   (d) General solution:   
   (e) ; ;   
     
   **(ii)** **Answer:** (a)   
   (b) ; we have a fundamental set of solutions.  
   (c) . is a solution of the complete equation.  
   (d) General solution:   
   (e) ; ;   
   **(iii)** **Answer:**   
   (a)   
   (b) ; we have a fundamental set of solutions.  
   (c) . is a solution of the complete equation.  
   (d) General solution:   
   (e) ; ;   
   **(iv)** **Answer:** (a)   
   (b) ; we have a fundamental set of solutions.  
   (c) . is a solution of the complete equation.  
   (d) General solution:   
   (e) ; ;   
   **(v)** **Answer:** (a)   
   (b) ; we have a fundamental set of solutions.  
   (c) . is a solution of the complete equation.  
   (d) General solution:   
   (e) ; ;
5. Let and be any two solutions of (3.18, p. 128) and let be the function whose value at is given by  
   . (Note that is the condition for and to form a fundamental set.) Write and use the fact that and satisfy (3.18) to show that . But (why?). Conclude that either is identically zero, i.e., for all , or is never zero.   
   **Answer:**  
   . If , then (3.18) is a first-order and not a second-order difference equation.  
      
    as claimed. Also as claimed, this implies that either is identically zero, i.e., for all , or is never zero.

# 3.3 general solution of the homogeneous equation (pp. 134 – 143)

1. Find the general solution of each of the following homogeneous difference equations:  
   **(a)** . **Answer:** . Thus,   
   **(b)** **Answer:** . Thus,  
      
   **(c)** **Answer:** . Thus,  
      
   **(d)** **Answer:** . Thus,   
   **(e)** **Answer:** . Thus,  
      
   **(f)** **Answer:** . Thus,
2. For each of the equations in Problem 1, find a particular solution satisfying the initial conditions and .  
   **(a)** . **Answer:** .   
      
   **(b)** **Answer:**   
   **(c)** **Answer:** .   
    .  
   **(d)** **Answer:** .   
   **(e)** **Answer:** . Since   
   **(f)** **Answer:** . and . . .
3. For each of the particular solutions obtained in Problem 2, find the first seven terms of the solution sequence . Conjecture as to the type of sequence obtained. (Review Section 2.5 first.) (Excel used for calculations.)  
   **(a)** . **Answer:** . Type D3, oscillates finitely.   
   **(b)** **Answer:** Type D1, diverges to   
   **(c)** **Answer:** Type D4, oscillates infinitely.   
   **(d)** **Answer:** (rounded to two digits). Type C3, bounded, monotone decreasing.   
   **(e)** **Answer:** Type D4, oscillates infinitely.  
   **(f)** **Answer:** Type D4, oscillates infinitely.
4. Write the difference equation (3.52, p. 141) in operator notation as , where is the displacement operator of the calculus of finite differences (see Section 1.3). If and are roots of the auxiliary equation (3.53, p. 141), show that (see Section 1.5) .  
   Hence the difference equation becomes  
   (3.58) . Let be a new function for which  
   (3.59) . Then (3.58) becomes .  
   Solve this first-order difference equation and thus show that is a solution. With this , show that (3.59) may be rewritten in the form .  
   Finally show (cf. 8 of Problems 2.4) that the solution of this equation is of the form   
   , and  
   .  
   **Answer:**  
   (3.52) . Recall: . Thus, we can write the terms of (3.52) as follows: and (3.52) as , as stated in this problem. The auxiliary equation (see p. 135) is If and are roots of the auxiliary equation, by definition , which is why we use the identity rather than the equal sign. Hence, the difference equation becomes  
   (3.58) . Let be a new function for which  
   (3.59) . Then (3.58) becomes , which has the solution  
   . Rewrite (3.59): or . Solve this, using (2.46, p. 69), which is part of 8 of Problems 2.4:  
    and   
   .

#### Note on (3.47), (3.48), and (3.49) on page 140.

Suppose and are complex conjugate roots of the auxiliary equation. We can write them in polar form:  
(3.47) and   
Assume that are complex conjugates: and .   
Apply de Moivre’s theorem: and   
Then, we can calculate   
(3.48)   
Define and and take the place of . and follow from the auxiliary equation. Then:  
(3.49) , with arbitrary constants.  
[Copied almost verbatim from Goldberg, p. 140]

# 3.4 Particular solutions of the complete equation (pp. 143 – 148)

1. Find particular solutions of the following difference equations by the method of undetermined coefficients:  
   **(a)** . **Answer:** Trial .  
   **(b)** . **Answer:** Trial .  
   **(c)** . **Answer:** Trial trial solution  
      
   **(d)** . **Answer:** Trial   
      
   **(e)** . **Answer:** Trial . Trial 2   
      
   **(f)** . **Answer:** Trial   
   **(g)** . **Answer:** Trial 1 – does not work. Trial 2 .  
   Trial 3   
      
   **(h)** . **Answer:** Trial
2. Find the general solutions of the difference equations in Problem 1. (First find the general solutions of the corresponding homogeneous equations and then use Theorem 3.6, p. 132.)  
   **(a)** .. **Answer:** General solution: .  
   .  
   **(b)** . . **Answer:** General solution: .  
   **(c)** . . **Answer:** General solution: .  
   .  
   **(d)** .. **Answer:** General solution: . .  
   **(e)** .. **Answer:** General solution:   
   . Find such that , which is the case for . By (3.49, p.140) .   
   **(f)** .. **Answer:** General solution: , .   
   .  
   **(g)** . . **Answer:** General solution:   
      
   **(h)** .. **Answer:** General solution:
3. Find the solutions of the difference equations in Problem 1 which satisfy the initial conditions and . (*Hint*: Find appropriate values for the arbitrary constants in the general solutions found in Problem 2.)  
   **(a)** . **Answer:** . , .   
   **(b)** . **Answer:** . , .   
   **(c)** . **Answer:** . and .   
   **(d)** . **Answer:** .   
   , , .   
   **(e)** . **Answer:** . .   
   **(f)** . **Answer:** .   
   **(g)** . **Answer:** .  
      
   **(h)** . **Answer:**
4. For the difference equation in Problem 1(a), find the solution which satisfies the initial conditions and . Note that your answer is the solution obtained in Problem 3 for the initial conditions and . Why is this so? **Answer:**  
    and .   
   The two solutions have in common.
5. Show that if is a solution of and is a solution of , then is a solution of . **Answer:**  
   By assumption:  and . Add those two equations:  
   . Thus, is a solution of , as claimed.

# Limiting Behavior of Solutions (pp. 148 – 153)

1. In 2 of Problems 3.3, particular solutions of six difference equations were obtained. (i) Determine the limiting behavior of each solution. (ii) Calculate for each difference equation and show that the condition is satisfied for only one of these equations.  
   **(a)** . **Answer:** (i) Oscillates between and .   
   (ii)   
   **(b)** . **Answer:** (i) Diverges to as . (ii)   
   **(c)** . **Answer:** (i) Oscillates infinitely .  
   (ii)   
   **(d)** . **Answer:** (ii) converges to as . (ii)   
   **(e)** . **Answer:** Since this sequence does not converge. It oscillates infinitely.  
      
   **(f)** . **Answer:** (i) Oscillates infinitely. (ii)
2. Consider the nonhomogeneous difference equations in 1 of Problems 3.4. The general solutions of the corresponding homogeneous equations were found in 2 of Problem 3.4. (i) Find the particular solutions of these homogeneous equations which satisfy the initial conditions and . (ii) Determine the limiting behavior of each solution.  
   In 3 of Problems 3.4, solutions of the complete equations were found subject to the same initial conditions. (iii) Determine the limiting behavior of the complete equation and compare with the behavior of the solution of the corresponding homogeneous equation.  
   **(a)** ; . **Answer:** (i) . . (ii) Monotone decreasing to . (iii) Same **(b)** . **Answer:** . (ii) Monotone decreasing to 0. (iii) Diverging to   
   **(c)** . **Answer:** (i) . (ii) Diverges to . (iii) Same  
   **(d)** . **Answer:** (i) (ii) Oscillates between . (iii) Diverges to .  
   **(e)** . **Answer:** (i) ; . . (ii) Oscillates infinitely. (iii) Same  
   **(f)** . **Answer:** (i) . (ii) Converges to 0. (iii) Converges to 2.  
   **(g)** **Answer:** (i) . (ii) Diverges to . (iii) Diverges to .  
   **(h)** ; . **Answer:** (i) . (ii) Diverges to . (iii) Diverges to .
3. Consider the equation , where is a positive constant. Solve the equation for and and show that the solutions are oscillatory, but with damped, finite, and infinite oscillations respectively. **Answer:**  
   :   
    and .  
   : and   
      
   : and

# Illustrative Examples from the Social Sciences (pp. 153 – 162)

1. Consider (3.65, p. 153) in the case where and . Show that the equation reduces to a nonhomogeneous first-order difference equation. Find the solution of which and discuss the behavior of the sequence. **Answer:**  
   (3.65) if and . This is a first-order nonhomogeneous difference equation, as claimed. Using Theorem 2.3 (p. 67), the solution of this difference equation is . For we have and converges to .
2. Consider (3.65) with and . Show that one obtains an “ever-increasing national income, eventually approaching a compound interest rate of growth,” except if for all . **Answer:**  
   . .   
   For the particular solution, try . Therefore, the solution is:  
   . If and and   
    for all in this special case.
3. A sequence of positive integers is determined by requiring that each term (after the first two) of the sequence is the sum of the two preceding terms. Let denote the th term . The values and are prescribed. (This sequence, with terms , is the so-called *Fibonacci sequence*.) Show that   
    so that satisfies the same difference equation as the function considered in Example 2. Find the solution subject to the given initial conditions and show that term number in the Fibonacci sequence is . **Answer:**  
      
    I  
   Then, , as claimed.
4. Consider (3.72, p. 156) in the special case , i.e., where both signals and take 1 unit of time for transmission over the channel. Show that and solve with the initial condition Show that the capacity of the channel is . **Answer:**  
   (3.72) . If then:  
   .   
   The capacity of the channel is defined by (3.76):
5. Consider (3.72, p. 156) in the special case , i.e., where both signals and take 2 units of time for transmission over the channel. Show that and solve with the initial conditions Show that the capacity of the channel is . **Answer:**. Rewrite:   
      
   Since and . If is even: ; if is odd:   
   . If is even: ; if is odd: . Hence
6. Follow the analysis in the text, starting with (3.77, p. 158), assuming and . If the initial values and are prescribed, show that the national income in period is given by  
    Calculate the values and note the damped oscillations. **Answer:**  
   (3.77) . If and .  
    . . and   
   . In this case we have . Thus,  
    . For the particular solution, try   
   The solution then is   
   .   
   . Make use of .  
   Check:   
    (Calculated using Excel). Graphically:
7. Using the data of Problem 6, show that (3.85, p. 159) reduces to . Assume and initial inventory level and use this equation and the values obtained in Problem 6 to calculate the inventory levels . Note the way in which inventory lags behind income. **Answer:**  
   (3.85) . Assuming and since   
    .  
   These results show that inventory still rises after income starts declining and begins rising again after income has already increased.  
   A graph of and as functions of follows on the next page.
8. In Example 3, show that if , the marginal propensity to consume is greater than , the income function of (3.77) does not oscillate. If initial values are excluded for which the arbitrary constants in the solution of the homogeneous equation become , show that the sequence diverges to in this case. [*Hint:* Both roots (3.79) of the auxiliary equation are positive and the larger is greater than .] **Answer:**  
   (3.77) . If , then   
    . Since for , and   
   The sequence diverges to .
9. Suppose level of sales in periods and are known and sales in period are to be estimated. Metzler defines the coefficient of expectation, , as follows: . Show **(a)** if , then a given level of sales in period is expected to be maintained in period ;   
   **(b)** if , sales in period are expected to differ from sales in period in precisely the same way as sales in period differed from sales in period ;   
   **(c)** if , an observed change of sales between periods and is expected to be temporary so that sales in period are expected to return to their level in period .  
   If sales equal in period and sales equal in period , calculate anticipated sales in period if , and . **Answer:**  
   **(a)**   
   **(b)**   
   **(c)**   
   **Assume** and   
   If , then by result (a),   
   If , then by result (b),   
   If , then ,   
   Thus,   
   If , then .   
   Thus,   
   If , then by result (c),

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  |  |  |  |  |  |
|  |  |  |  |  |  |

1. In 13 Problem 2.7, the difference equation , with , was derived for the amount of an increasing annuity. Obtain the solution given there by using Theorem 3.3 (p. 125). (Find a particular solution of the complete equation by the method of undetermined coefficients, pp. 143-147). **Answer:**   
   Write the homogeneous difference equation: .   
   Particular solution: ;   
   ; , which is the same as the solution in 13 Problem 2.7.
2. Suppose , where and are positive constants. Find the general solution and determine its limiting behavior if **(a)** ; **(b)** ; **(c)** . (See 8 of Problems 2.8 and footnote 15 cited there.) **Answer:**  
   Write difference equation: .  
      
   Particular solution: since   
   General solution:   
   **(a)** : and   
   ; oscillates finitely.  
   **(b)** : oscillates infinitely.  
   **(c)** :   
    or radians  
    damped oscillation around its limit .
3. Consider the difference equation , where and are positive constants and is warranted income in period . By studying the roots of the auxiliary equation, show that “For this gives an explosive time path for warranted income…, but for warranted income behaves cyclically.” (Cf. 6 of Problems 2.8. Quote from R.R. Bush and J.W.M. Whiting 1953, *Journal of Abnormal and Social Psychology*.) **Answer:**  
   Auxiliary equation:   
   : Choose “explosive growth”  
   Note that if then since , but and so that we always have explosive growth.  
   : In this case, and is of the form , which means that the sequence oscillates, that is, behaves cyclically.   
   ; thus which suggests that makes no economic sense. if . Therefore:   
   : infinitely oscillatory; : finitely oscillatory; : damped oscillatory. for all , which makes no economic sense.
4. Consider the difference equation , where and are constants. If   
    and are prescribed, show that , where  
   , . **Answer:**  
   Auxiliary equation:   
   . Then,   
    ; note that   
      
   Therefore:

# 3.7 The general case of order n (pp. 163 – 167)

1. The following difference equations have been constructed so that their auxiliary equations have at least one integral root and hence can easily be factored. Find the remaining root and write the general solutions of each difference equation. (I used an online polynomial equation solver)  
   **(a)** **Answer:**   
      
   . It turns out that .   
   General solution: **(b)** **Answer:**   
   . Particular solution:   
   General solution:   
   **(c)** **Answer:** try and use synthetic division:

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| 1 | 1 | -3 | 4 | -2 |
|  |  | 1 | -2 | 2 |
|  | 1 | -2 | 2 | 0 |

is a root.   
   
**(d)** **Answer:** . is a fairly obvious solution. Use synthetic division:

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| 1 | 1 | -2 | 2 | -2 | 1 |
|  |  | 1 | -1 | 1 | -1 |
|  | 1 | -1 | 1 | -1 | 0 |

Hence . is also a fairly obvious solution.

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| 1 | 1 | -1 | 1 | -1 |
|  |  | 1 | 0 | 1 |
|  | 1 | 0 | 1 | 0 |

.  
.   
**(e)** **Answer:** . We recognize this equation as   
General solution:

1. Consider the difference equation where is a constant greater the . Show that is a root of the auxiliary equation and thus obtain the auxiliary equation in the form  
   . Use Descartes’ rule of signs to show that the auxiliary equation can have at most one positive root in addition to . Show that the bracketed quantity is negative when and positive for some sufficiently large value of and conclude that this additional root exists and is greater than . What then is the nature of the solution sequence ? **Answer:**  
   **Auxiliary equation:** . It is easy to see that is a solution: .  
   Use synthetic division with :

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  |  |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |  |

Therefore: as claimed.  
**:** because .  
**:**   
. If , then .

. There is only one sign change in this equation. From Descartes’ rule of signs, we conclude that the bracketed term has at most one positive root and the whole difference equation therefore has two. The second positive root must be greater than .

# Selected topics

# Equlibrium and stability (pp. 169 – 176)

1. If , show that an equilibrium value of is given by if . Prove, using Theorem 2.7 (p. 86), that is a necessary and sufficient condition for this equilibrium to be stable. **Answer:** We knowthat the solution of the difference equation when is: (Theorem 2.2, p. 64.) This solution can be rewritten: . Clearly, is an equilibrium because if  
    then for all . This is a stable equilibrium if . This is the case iff or . This is confirmed by Theorem 2.7.  
   Notice that if , then .
2. **(a)** Verify that the values and satisfy both (4.16) and (4.17) and conclude that in this case the solution of (4.14) undergoes damped oscillations toward the stable equilibrium value .  
   **(b)** Verify that and do not satisfy the stability conditions (4.16) but do satisfy (4.17). Conclude that now undergoes infinite oscillations. (Review the discussion in Example 1 or Section 3.6.) **Answer:**  
    **(a)** (4.14, p. 173) . The stability conditions (4.16) are and . If and , this condition is satisfied. Condition (4.17) is . If and , then   
   . Recall that . In this case, .  
   **(b)** If and , then but and does not satisfy (4.16). In this case,  
    and (4.17) is met. Therefore, undergoes infinite oscillations, as claimed.  
   (See Example on page 173.)
3. If the second-order difference equation is written in the form with and constants, show that the stability conditions (4.13) may be written .  
   **Answer:**  
   Use (4.13; part of Theorem 4.2, p. 172): .   
   From the first inequality: . From the second inequality:   
   Therefore: as claimed. follows from the third inequality.
4. Consider the difference equation   
    where all coefficients are positive constants with .  
   Find the equilibrium value of and apply the conditions (4.13) to show that the equilibrium is stable.  
   **Answer:**If an **equilibrium** exists, then   
   Hence, . This is the equilibrium value.  
   To check for **stability** of the equilibrium use (4.13).   
   . Since . The first inequality in (4.13) is satisfied.   
    because if we subtract we obtain . Therefore, the second inequality is also satisfied.   
    because both terms are positive. Therefore, the third inequality is satisfied and is a stable equilibrium.
5. In (4.19) assume . **(a)** Show, by verifying conditions (4.13), (b) Show that the moving equilibrium is of the form . **Answer:**  
   (4.19)   
   **(a)** Equilibrium conditions: . By assumption, this condition is met:  
      
   . By the same assumption, this condition is also met:   
    is met because   
   (b) If an equilibrium exists: .
6. In (4.19) suppose (i.e., consumption depends only on income one period before.) Solve the difference equation and show that and hence deduce that the moving equilibrium is of the form if . **Answer:**  
   (4.19) when is a first-order difference equation. The general solution is:  
   .   
   If an equilibrium exists: if .  
   Hence, and if .   
   Since

# First-order equations and Cobweb cycles (pp. 176 – 184)

1. Find the general solution of each difference equation and then find the particular solution for which .  
   **(a)** . **Answer:**   
   **(b)** **Answer:** General solution:   
   **(c)** . **Answer:** .  
   . Reminder:   
   **(d)** . **Answer:** Same process as before with in place of . Hence: . . Therefore: .
2. In the difference equation  make the substitution and show that the equation becomes , from which or . Hence find the two solutions and given by and . These are two different solutions, both assuming the same initial value . Why does this fact not violate the uniqueness theorem of Section 2.3? **Answer:**  
    as claimed and also or . and .   
   It does not violate the uniqueness theorem of Section 2.3 because the difference equation in this problem is not linear. Therefore, the uniqueness theorem does not apply to this equation.
3. If and is prescribed, show that . Conclude that the sequence is an unending repetition of only three values. **Answer:**  
   . Since , etc. In general, .
4. Show that the difference equation becomes a linear equation  if  
   . In the special case , find and show that . Show that the sequence converges with limit 2. **Answer:**  
   . Hence, , as claimed.  
   if , then . To find the general solution: . Therefore, we have . Thus: and . Use this solution to find . For . Hence, .