**Solutions to Problems in:**

**John Hoag, 2008.**

**Calculus and Techniques of Optimization with Microeconomic Applications**

**World Scientific**

PV Schaeffer

**Section I: Sets and Functions**

# Exercise 1

Suppose and . Write out .

**Answer:**

# Exercise 2

Which of the following are function? For each function, what is the domain and the range?

**Answer:** This is not a function because the number 4 is associate first with 1, and then with 2. That is, it violates definition 1.10.

**Answer:** This is a function. is both the domain and the range of this function.

**Answer:** This is not a function because it is not defined for negative values of .

**Answer:** This is not a function because it is not defined for .

# Exercise 3

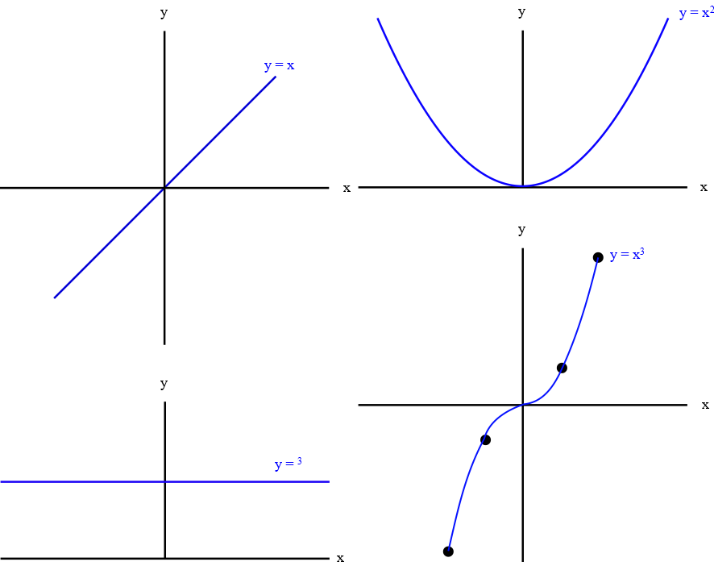
Which of the following are one to one? (Hint: draw a graph.)

**Answer:** This is one to one. (It is a line through the origin.)

**Answer:** This is not one to one because, for example, if and also if .

**Answer:** This is not a function because it is not defined for negative values of .

**Answer:** This is not a function because it is not defined for .



# Exercise 4

Suppose that we define a function with the domain of the positive integers. For example, . Is this a function (again remember that the domain is the set of integers, not the real numbers)? Is this function one to one?

**Answer:** Yes, this function satisfies definition 1.10. It is one to one.

# Exercise 5

Let be defined on the positive integers. Suppose . Is this a function? Is this function one to one?

**Answer:** Yes, this function satisfies definition 1.10. It is one to one

**Section III: Matrix Algebra**

# Exercise 1

In the matrix shown above, write the elements of column j. Write the elements of row j.

**Answer:** Column elements: Row elements:

# Exercise 2

Unpack III.7

**Answer:** If , then

**.**

# Exercise 3

Find the product of the following real numbers and matrices.

1. . **Answer:**
2. . **Answer:** .
3. . **Answer:** .

# Exercise 4

Add the following matrices.

1. . cannot be calculated because is a matrix and is a matrix.
2. . .
3. .
4. .
5. Use the matrices in b and find .
6. Use the matrices in c and find .
7. .

# Exercise 5

Suppose that and are all matrices. How would we find ? Would How would you know?

**Answer:**

* Call the sum of the three matrices . Then .
* because the order in which we add , and does not matter.

# Exercise 6

Write out the following sums.

# Exercise 7

1. From example c, find the product of . is a matrix and is a matrix. While the product of is defined, the product of is not.
2. From example d, find the product of . .

.

1. Find .
2. For and in Exercise 6c, find This is not defined: is a matrix and is a matrix. While the product of is defined, the product of is not.

Find

.

1. For the and in e, find

1. ; ;

.

.

Hence: .

1. and The product is

# Exercise 8

Consider the matrix . Multiply this matrix by the scalar and call the outcome . What do you get? Now multiply by the matrix , where . What do you get? How is this product related to ? Explain why these two should be so related.

**Answer:**

.

For the second part: .

Since and we can see from the above that every element of is being multiplied by 2. Multiplying by the scalar of by the matrix yields the same result.

# Exercise 9

1. Find the product of by first finding and then multiplying that product by . (From here on we are dropping the use of to indicate matrix multiplication).

.

.

.

1. Now multiply and multiply that product by (i.e., multiply times the product – be careful of the order of multiplication).

Introducing Falk’s Triangle:

|  |  |  |
| --- | --- | --- |
|  |  |  |
|  |  |

|  |  |  |
| --- | --- | --- |
|  |  |  |
|  |  |

The order of multiplication matters!

1. Multiply the following matrices.

. This is a quadratic function.

. This is also a quadratic function.

1. Show that for any matrices , and that .

**Answer:**

The typical element of ; denote this typical element . Then the typical element of .

.

The typical element of ; denote this typical element . Then the typical element of .

.

This property is called the associative property.

# Exercise 10

III.12. Theorem: Suppose that is and is . Then

Prove this proposition by computing both sides of the equation.

**Answer:**

.

. This shows that .

# Exercise 11

Suppose that you have two matrices and shown below.

1. Write out the product .

.

1. Write out

**Answer:**

1. Write the transpose of and the transpose of .

**Answer:**

1. Compute . How does this relate to the product in part b?

**Answer:** . It is the same. This shows that .

# Exercise 12

Suppose we have the following matrix.

.

1. Write the transpose of .
2. What is in ? . What is in ’? .
3. No, is not symmetric. Symmetry requires that in in

# Exercise 13

Find the determinants of the following matrices.

1. .
2. .
3. .
4. .

# Exercise 14

Find the determinants of the following matrices.

.

1. .
2. .
3. For , show by computing both determinants, that the determinant of and the determinant of are equal.

.

;

.

This shows that .

1. Find the determinant of . Now for , find the determinant of

.

.

1. Find the determinant of . Now for , find the determinant of

.

.

1. Find the determinant of . Now for , find the determinant of

.

.

.

**Extra – not from book:**

.

.

.

# Exercise 15

Suppose that we have a matrix and want to find a fourth-order minor. In the definition of the order minor, what is the value of ? **Answer:** .

How many rows would we have to cross out? How many columns? **Answer:** .

What would be the dimension of the matrix whose determinant we would want to find to this minor? **Answer:** .

# Exercise 16

If we have a matrix, what are the first-order minors?

**Answer:** ; ; ; ; ; ; ; ;.

How many first-order minors are there for this matrix? **Answer:** There are minors because we have three choices of the row to cross out and three choices of the column to cross out. Graphically shown on next page.

How many rows and how many columns do we cross out? **Answer:** 1 row and 1 column.

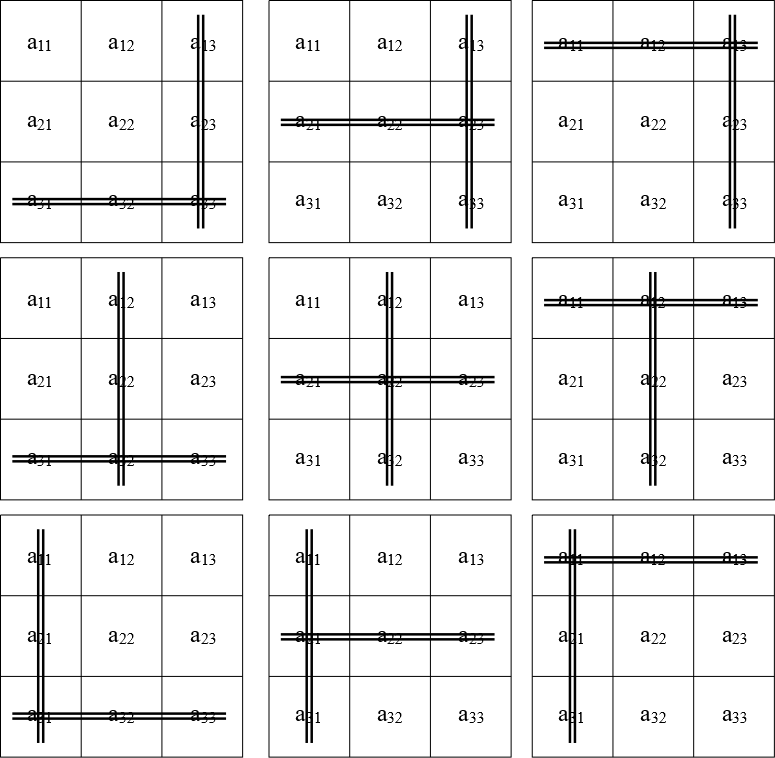
For the same matrix, for the second-order minors, how many rows and how many columns do we cross out? **Answer:** 2 rows and 2 columns,

How many rows and how many columns are left? **Answer:** One of each.

How many second-order minors are there in this matrix? **Answer:** Nine; each element is a minor.

Is there are third-order minor? **Answer:** No.

Number of first-order minors in a matrix:

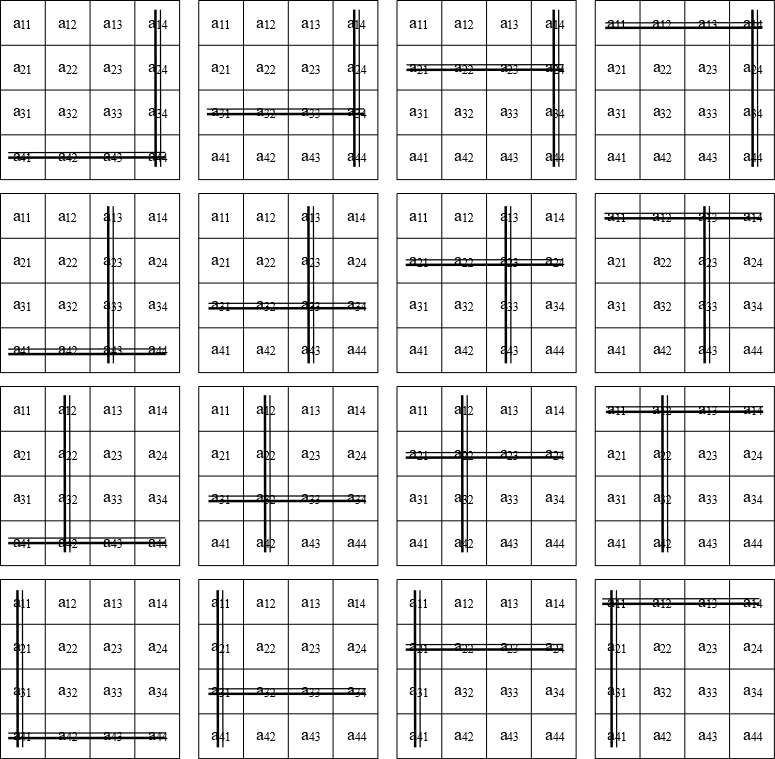


# Exercise 17

Given a matrix , find all third-order minors of . Find all first-order minors of .

**Answer:** The third-order minors are all the sixteen elements of the matrix. . There are also 16 first-order minors. We should them graphically on the next page.

Number of first-order minors in a matrix:



# Exercise 18

For the matrix , find all of the minors of all sizes.

**First-order minors:**

; ; ; ;

**Second-order minors:**

; ; ; ;

# Exercise 19

Find all the minors for the following matrix. Which minors are the principal minors?

**First-order minors:** ; ; ; ; ; ; ; ;. There are three principal minors; they are highlighted.

**Second-order minors:** ; ; ; ; ; they are all principal minors.

# Exercise 20

For a matrix find all third-order principal minors of .

**Answer:**

# Exercise 21

How many principal minors are there for a matrix?

|  |  |
| --- | --- |
| First-order: |  |
| Second-order: |
| Third-order: |

# Exercise 22

Suppose . Find the determinant of . Now write . Find the determinant of . How are these two determinants related?

**Answer:** . . .

# Exercise 23

Suppose . Write the determinant of . Now write . Find the determinant of . How are these two determinants related? Based on what you have done, can you see a way to show that for a matrix, . What must be true generally for and

**Answer:** .

; .

This shows that .

# Exercise 24

Suppose is symmetric and square. What can you say about the minors of the elements and ? What if is not symmetric?

**is symmetric:** In this case . Use .

The minor of . The minor of . If we calculate the determinants, they are the same. Thus, . This likely to be true for all symmetric square matrices, but I do not know how to prove the general case.

**is not symmetric:** Use. The minor of .

The minor of . The two minors are not identical.

# Exercise 25

Find the cofactors for all elements of the matrix in Exercise 17.

**Answer:** We will only show the first four first-order cofactors (first-order signed minor) from the second row.

; ; ;

# Exercise 26

Supposes that A is symmetric and square. How will the cofactor of element be related to the cofactor of element ? Explain.

**Answer:** We have already shown that the affiliated minors are the same. Since , their cofactors are the same, too.

# Exercise 27

Find the determinants of the following matrices.

1. .
2. .
3. .
4. and .
5. .
6. .
7. .
8. . .
9. .

.

How are the two related? **Answer:** I do not know.

# Exercise 28 & 29

Suppose is an matrix whose determinant is not . Suppose we multiply one row and one column by a constant, . What happens to the determinant as a result? Explain.

**Answer:** We know that . From this follows that if we multiply only one row by , then the determinant of this new matrix (call it ) will be equal to . If now we multiply one column of by the constant , then the new matrix that results (call it ) will be equal to .

Hence, if we multiply one row and one column by a constant, then the new matrix that results will have the determinant .

# Exercise 30

Suppose is an matrix whose determinant is not . Suppose we multiply every element of by a constant, . What happens to the determinant as a result?

**Answer:** The answer follows from Exercise 28 & 29. If we multiply, say, one row by , the new matrix has determinant . If we now multiply the second row, as well, then the new determinant is . If we then multiply the third row, the determinant is . If we keep going until we multiply the last () row, we will have multiplied all elements by and the new determinant is .

# Exercise 31

For the following matrices, find the determinant. Then multiply the indicated row and add that row to row 1 to form a new matrix. The row you multiplied does not change. Find the determinant of the new matrix.

Modified

.

.

Modified

.

# Exercise 32

Where in the proof (III.26. Theorem) did we use Alien Cofactors? Where did we use Laplace (expansion)?

**Answer:** We used both of them in the second step, after multiplying the matrix and the matrix of its cofactors, .

# Exercise 33

Find the inverse for the matrices in Exercise 13.

1. The inverse of the identity matrix is the identity matrix. We already know that Hence, .
2. . To check the result, multiply .
3. . . .

.

1. . . .

.

# Exercise 34

Show that if is , as shown, then the inverse of is , where .

**Answer:** We just showed this in Exercise 33.d., where .

# Exercise 35

Find the inverse of the following matrices.

1. .
2. .

.

Check by calculating .

1. The inverse of a identity matrix is a identity matrix (since and the only non-zero cofactors are

.

Check by calculating .

.

Check by calculating .

.

Check by calculating .

# Exercise 36

We know if we multiply a row by a constant and add it to another row, we do not change the determinant. Suppose we multiply a row by a constant and add that to another row. Will this change the inverse? Use the matrix in Exercise 34.c. First find the determinant of this matrix. Find the inverse of this matrix. Now multiply row 2 in the matrix in 34.c. by 2 and add to the first row of that matrix. Find the determinant of this new matrix. Is it the same as before you made the change? Find the inverse of the new matrix. Is it the same as before you made the change?

**Answer:** We use the matrix from Exercise 35.c. because there is no part c. in Exercise 34.

.

Modified . Modified , the same as before.

Modified .

Check:

**Conclusion:** The determinants of the original and the modified matrix are the same, but the inverses differ.

# Exercise 37

Suppose that is symmetric, square, and has a nonzero determinant. How is related to ? Is symmetric in this case? Explain.

**Answer:** Since is symmetric, . Hence, . We continue as follows, using : . Since we rewrite the highlighted part: , the last step by the symmetry of . This shows that and since .

**Conclusion:** We showed that the inverse of a symmetric matrix is also symmetric.

# Exercise 38

Show that for and matrices, that .

**Answer:**

Assume that is the inverse of . Then . We have used the associative property of multiplication and have shown that .

**Section IV: Linear Algebra**

# Exercise 1

Why can’t we multiply the vectors and ?

**Answer:** They both are matrices. They do not match up.

# Exercise 2

Multiply the following vectors by the given scalar.

1. times
2. times
3. times

# Exercise 3

Let and . Use graphical methods to find (We will not use graphical methods because it is difficult to draw the results accurately in three dimensions.):

1. and . and .
2. .
3. .

# Exercise 4

Unpack definition IV.3 (linear combination)

**Answer:** A linear combination is a scaled sum of vectors in space. This means each vector has components and the s are scalars. The vector that results from the linear combination also has components.

# Exercise 5

Write out a linear combination of the vectors , where and .

**Answer:**

# Exercise 6

Write out a general linear combination of the vectors in Exercise 5 just using the s, but not using specific values of .

**Answer:**

# Example, pp. 42-43

Are these vectors linearly dependent or independent? ;

Answer: Note these are the same vectors as in Exercise 5 and 6. We have already shown that

Step 1: ; ; and . If we can find s not all zero, we have shown linear dependence.

Step 2: Solve the last equation for .

Step 3: Use this value for in the first two equations: and .

Step 4: If we add these equations we obtain a value for : ; therefore, , as well, and

**Conclusion:** Since we only found solutions for the s that are all zero, the vectors are linearly independent.

# Exercise 8

Does have an inverse? How do you know? What is the inverse?

**Answer:** We know how to find the inverse, if it exists: . Check: .

# Exercise 9

How did we go from to 3 and ?

**Answer:** We added the two vectors to obtain .

# Exercise 10

Suppose that we have two vectors and . Show that if and are linearly dependent, then for some nonzero value of .

**Answer:** This follows directly from the definition of linear dependence.

# Exercise 11

Which of the following sets of vectors are linearly independent? Linearly dependent? Use definition to find out.

1. .

**Answer:**

Step 1: . Calculate 2

Step 2: Calculate

Step 3: The only possible solution is Using this, it follows from that , as well.

**Conclusion:** The vectors are linearly independent.

1. .

**Answer:** Write this as an equation system: ; find the inverse of the matrix. If it exists, the vectors are linearly independent. First calculate . This means that the inverse exists. The vectors are linearly independent.

1. .

**Answer:**

Step 1: . From .

Step 2: From and from .

Step 3: The only possible solution is

**Conclusion:** The vectors are linearly independent.

1. .

**Answer:**

Step 1:

Step 2: .

Step 3: Substitute into .

Step 4: Let These nonzero values satisfy . and .

**Conclusion:** The vectors are linearly dependent.

1. .

**Answer:**

Step 1:

Step 2: From . Substitute into .

Step 3: Use Use .

Step 4: This means .

**Conclusion:** The vectors are linearly independent.

1. .

**Answer:**

Step 1:

Step 2: . Substitute into .

Step 3: Let . Then and . This does not satisfy and Only satisfies all three equations.

**Conclusion:** The vectors are linearly independent.

1. .

**Answer:** We can write four equations, but we have only three unknowns .

Step 1: .

Step 2: From . Substitute into .

Step 3: Combine the results from step 2: . Set .

This does not satisfy and Only satisfies all three equations.

**Conclusion:** The vectors are linearly independent.

1. .

**Answer:** The independence of these four equations is “obvious.”

1. .

**Answer:**

Step 1:

Step 2: . Only satisfies all three equations.

**Conclusion:** The vectors are linearly independent.

1. .

**Answer:** We can write four equations, but we have only three unknowns .

Step 1:

Step 2:

Step 3: Let . Using this result, find by using and then checking whether the result is true for all three equations. From .

Step 4: . We found a nonzero solution.

**Conclusion:** The vectors are linearly dependent.

1. .

**Answer:** . From this and the first equation . Only satisfies the two equations.

**Conclusion:** The vectors are linearly independent.

# Exercise 12

Suppose we have vectors, of elements each with . Show that the vectors are linearly independent if they have the following form.

**Answer:** cannot be written as a linear combination of the other vectors, because their first components are all zero.

cannot be written as a linear combination of the other vectors, because the vectors coming after vector 2 have their second components all zero, while the first vector has a nonzero first component.

, cannot be written as a linear combination of the other vectors that follow because their kth components are all zero, while the preceding vectors have positive components where the kth vector’s components are zero.

cannot be written as a linear combination of the preceding vectors because they have positive components where the mth vector’s components are zero.

# Exercise 13

is a linear transformation where is an matrix and and . Show that for this linear transformation.

**Answer:** We use properties of matrices. First, . The first equality follows from the associative property of matrix multiplication, and the second equality follows because are scalars.

# Exercise 14

Explain why the rank of an matrix must be .

**Answer:** I cannot prove it. However, it makes intuitive sense. Assume that that and assume that is the number of linearly independent columns. Thus, each column can be expressed as a linear combination of at most other columns in the matrix. But since together the columns also include all rows, it means that only of them can be linearly independent. (Read the remark in Hoag, pp. 50-51.)

# Exercise 15

Find the rank of the following matrices.

1. .
2. .
3. The first row can be obtained by multiplying the second row by

# Exercise 16

Show that if are linearly independent vectors from with , then we cannot write as a linear combination of . How does this problem change if ?

**Answer:** We cannot write as a linear combination of because are linearly independent vectors. If , then at most of the vectors from can be linearly independent. Therefore, at least one can be written as linear combinations of a subset of these vectors.

# Exercise 17

Determine which of the following systems have solutions and whether the solutions are unique.

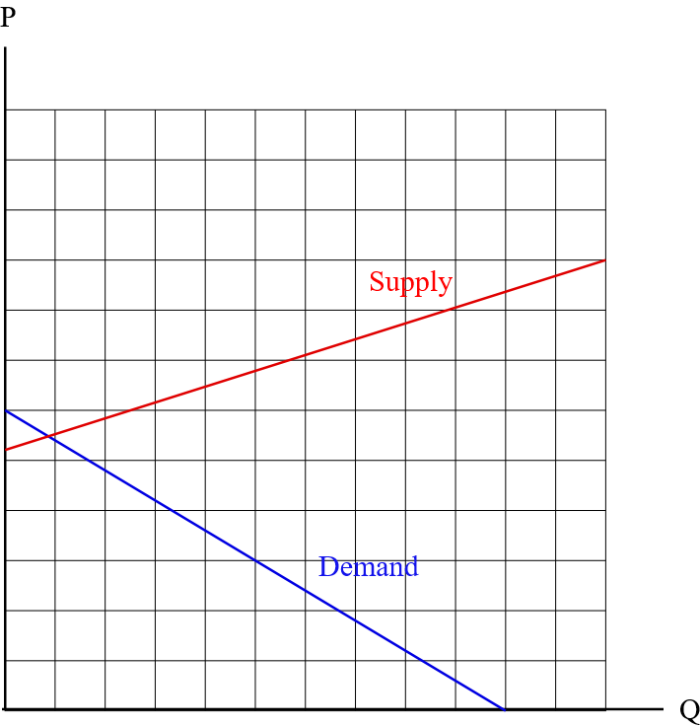
1. Then,

;

1. ;
2. Infinitely many solutions.

# Exercise 18

Suppose we have the following demand and supply curves. Find the equilibrium output and price using the methods developed above.



1. For the following problem, there are two good, and ; they have inter-related demands. Find the quantities of and and the prices and that cause both markets to clear simultaneously.

. All number are rounded to the second digit.

I used <http://matrix.reshish.com/inverCalculation.php> (January 5, 2018) for matrix inversion.

# Exercise 19 (Cramer’s Rule)

Cramer’s Rule. Suppose that we have the following square matric system, . The vector and . Suppose that we want to solve for one element of the vector, . Prove Cramer’s Rule.

**The proof** is from planetmath.org: Let be the column of . Let , where is an vector with all zeros except a one (1) as its component. , where is column. Use the property that determinant multiplication . We can calculate by expansion. Since is the identity matrix except where replaces the column of the identity matrix we have , where is the identity matrix.

Since , it follows that . This proves Cramer’s Rule.

(Source: <http://planetmath.org/proofofcramersrule>, January 5, 2018)

# Exercise 20

Suppose we have where is a matrix and is . Suppose that the determinant of What does this tell us about these lines? What does it tell us about the volume generated by these three vectors?

**Answer:** The vectors generate zero volume. All three either lie on the same plane or along the same line.

# Exercise 21

1. Write the equation of the straight line through the points and .

**Answer:**

|  |  |
| --- | --- |
|  | A general linear equation in two dimensions has the form . In this case:  and ; subtract the second equation from the first: , which agrees with what the graph shows. Now use this result in the first equation to obtain Check this result in the graph as well.  **Equation:** . |

1. Write the equation of the straight line through and .

**Answer:** A general linear equation in three dimensions has the form . In this case:

and . Calculate . Substitute this result into . **Equation:** .

1. Write the equation of the straight line through and .

**Answer:** A general linear equation in three dimensions has the form . In this case:

and . Calculate . Substitute into   
Substitute into

**Equation:**

1. Write the equation of the plane through the points and .

**Answer:** A general linear equation in three dimensions has the form . In this case:

; and . Calculate . Add .

Use ; .

**Equation:** .

# Exercise 22

This exercise seems trivial. I suspect it is meant as a step towards the next section (Section V. Calculus of One Variable).

**Section V: Calculus of One Variable**

# Exercise 1

The proof of Theorem V.3 (p. 72) says that part a and b are obvious. Explain, then, why they are true.

**Answer:**

1. . For any , by definition.
2. Only if all components of are zero can The only point that has this property is the origin.

# Exercise 2: Cauchy-Schwartz Inequality

The above proof relies upon the fact that Why is this inequality true? (Hint: This is the Cauchy-Schwartz inequality) take where and are scalars where and . Clearly Replace with , and replace and with the definitions given.

**Answer:**

if and the inequality is “trivially” true if ; .

**Alternative Proof** from <http://rgmia.org/papers/v12e/Cauchy-Schwarzinequality.pdf>, 1/9/2018:

. We divide both terms by 2 and move the second term to the RHS:

.

# Exercise 3

Are the following distance functions for ? Check to see if each element of the definition is satisfied. Show your work.

1. ; ; by part c of Theorem V.3. This is a distance function.
2. ; Therefore, this is not a distance function.
3. is not non-negative. If . Therefore, this is not a distance function.
4. ; ; . This is a distance function.
5. ; ;

# Exercise 4

In the following example, why are and both limit points of the set? **Example:** For and and and constants, .

**Answer:** There exists . The same can be shown for .

In the following example, why is the set not closed? What point is a limit point that is not in the set? Explain why it is a limit point. **Example:** For , and and and constants, .

**Answer:** There exists and . Hence both and are limit points. However, limit point is not in the set.

# Exercise 5

Give an example of a set in with exactly one limit point. Defend your answer. Hint: look at the example before V.9 and Example d before Exercise 4. Now explain why a set with a finite number of points must be closed. How many limit points does this set have? Does it contain all its limit points?

**Answer:** . In this set is the only limit point. It does not contain its limit point.

# Exercise 6

Prove that if is a closed set then the complement of is open. Suppose

**Answer:** Let be the complement of . Suppose that is open and that is a limit point of . Then contains some points in ( because is a limit point. Therefore is not an interior point of and thus . Thus contains its limit points and is closed.

Conversely, suppose is closed and choose and hence . But then, is an interior point of and cannot be a limit point of . Thus contains its limit points.

# Exercise 7

Which of the following sets are closed and bounded? Closed? Bounded?

1. The set positive constants. **Answer:** This set closed and bounded. The fact that is closed is obvious and it is bounded because cannot exceed and cannot exceed .
2. The set . **Answer:** This set is unbounded because the there is no limit to how large can be. It is closed because it contains its limit points ( is its only limit point).
3. The set positive constants. **Answer:** This set not closed and unbounded. Unbounded is easy to see because and/or can be any size. The fact that it is not a closed set follows from that observation that no limits for and as long as they satisfy .

# Exercise 8

Translate the following phrase from Definition V.13. into English. “If there is a number

**Answer:** If such a number exists, then it is at least as large as any number in . There is no number in larger than . Note that it is not required for to be in .

# Exercise 9

Find the least upper bound for the following sets.

1. **Answer:** No point in this set is as large as 55, which is the LUB.
2. **Answer:** No point in this set is larger than 55, which is the LUB.
3. **Answer:** No point in this set is larger than , which is the LUB.
4. **Answer:** This set has no LUB. To see this, assume that is the LUB. But there exists a positive integer, for example .

# Exercise 10

For and real numbers, show that .

**Answer:** Since , .

# Exercise 11

In the following graph (see Hoag, p. 85), suppose that the candidate for the limit in the example on pp. 84-85 is . Choose an smaller than . Can you find a that will keep you with that of ? Explain.

**Answer:**

|  |  |
| --- | --- |
|  | No matter what value of we choose, the neighborhood will include values . But for those values are not in . This means that B is not a limit for as . |

# Exercise 12

Suppose that the candidate for the limit in Exercise 11 is halfway between and at . Choose an smaller than . Can you find a that will keep you within that of ? Explain.

**Answer:**

No matter what value of we choose, the neighborhood will include values and . But for those values are not in . and This means that is not a limit for as .

# Exercise 13: Properties of limits

Prove limit rules 1, 4, and 5

**Rule 1:**

**Proof:**

Let and find . Because the limit for exists, there is a . Similarly, because the limit B for exists, there is a .

Choose . Then, if . This proves that .

Source: <http://www.math.uakron.edu/~dpstory/tutorial/c1/c1l_tp.pdf>

**Rule 4:** .

**Proof:**

Let and find . Choose .

Rule 5: .

**Proof:** Let and find . Choose .

# Exercise 14

Find the following limits

1. does not seem to have limit as . We illustrate this with the three graphs on the next page, showing the numerator, denominator, and their ratio (“function”).

|  |  |
| --- | --- |
|  |  |
|  | |

# Exercise 15

Translate “into words.

**Answer:**

For all positive numbers , there exists a real number such that if is larger than the distance between the value of the function and its limit (the limit when goes towards infinity) is less than .

# Exercise 16

Translate this statement into words: “We say that

**Answer:**

The limit of the function as approaches is infinite if and only if for all positive there exists a positive such that if the distance between and is less than , then the value of the function exceeds .

# Exercise 17

Show that is a limit point in the second remark on page 88.

**Answer:**

Apply Definition V.17. Choose arbitrarily large. Note that in this case Choose any . For any such .

# Exercise 18

Rewrite the definition of continuity in terms of the limit. Then rewrite the definition using the notation of mathematics. See the remark after V.16.

**Answer:**

The definition says that for all there exists a such that if then . This implies that if is a limit point of , then is the limit of .

is continuous at if then .

# Exercise 19

Use the epsilon-delta argument to show that the limit for in the example on pp. 90-91 cannot be either 11 or 20 as approaches 10. Why were these two numbers chosen as possible values for the limit? Based on your argument explain why there is no limit for this function as approaches 10.

**Answer:**

Assume the limit is 11. Choose . For no and can we find .

Assume the limit is 20: Choose . For no and can we find .

# Exercise 20

Use the epsilon-delta argument to show that the limit for as approaches is .

**Answer:**

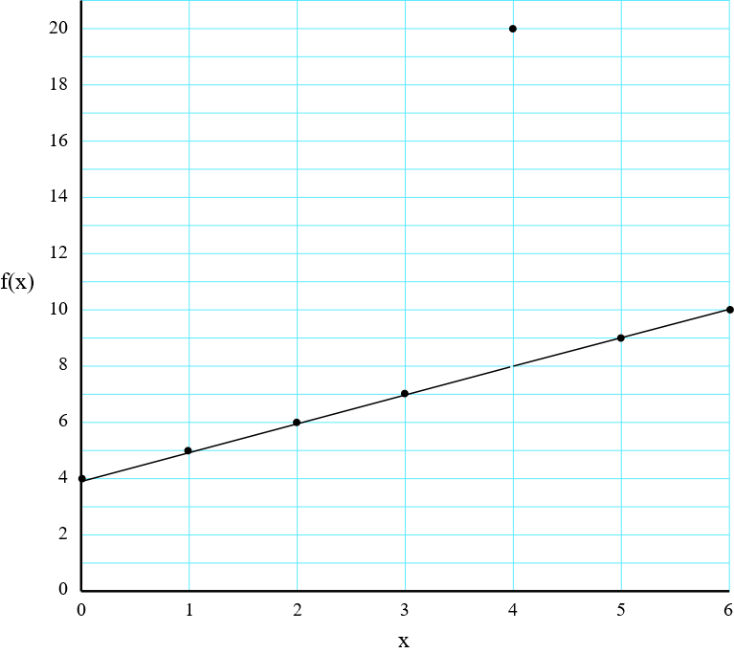
Choose an arbitrary . Since this means that .

# Exercise 21

Draw the graph of

**Answer:**

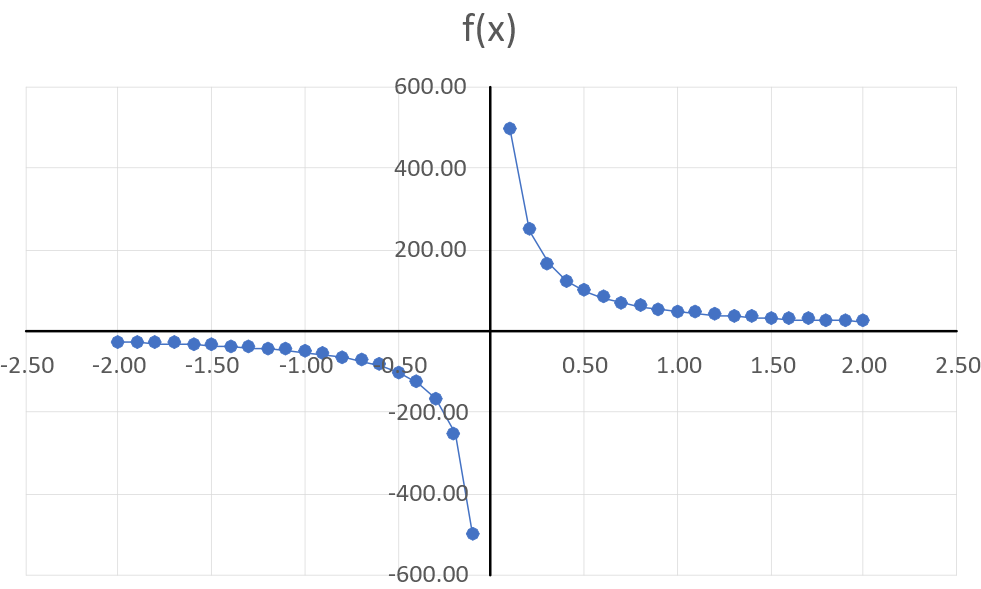
The graph is on the next page



# Exercise 22

Graph

**Answer:**



# Exercise 23

Show that if and are continuous, then is also continuous.

**Answer:**

. Similarly, Then .

# Exercise 24

Show that if and are continuous, then is also continuous. (HINT: use the fact that the limit of the product is the product of the limits.)

**Answer:**

Let Then . Thus, such that if . This shows that is continuous.

**Note:** I think this is correct, but I am not 100% sure. Therefore, I am offering an alternative proof which I found at <https://www.math.auckland.ac.nz/class255/08s1/01s2/H5.pdf>.

such that for is defined as follows: Without loss of generality, assume In addition, . Therefore,

.

# Exercise 25

Show that if then is continuous.

**Answer:**

Choose an Then, if .

# Exercise 26

Show that is continuous. (HINT: how can you use Exercises 24 and 25 to help?)

**Answer:**

Let and . Both functions are continuous (Exercise 25). The function is continuous (Exercise 24).

# Exercise 27

Show thatif then is continuous.

**Answer:**

Choose an Then, if .

# Exercise 28

Show thatif and g then is continuous.

**Answer:**

We have shown that is continuous (Exercise 27) and that g is also continuous (Exercise 26). Then is continuous (Exercise 24).

# Exercise 29

Show that if the limit as approaches from the left exists and is , and if the limit as approaches from the right exists and is , then the limit as approaches is .

**Answer:**

Limit from the Right: .

Limit from the Left: .

Together the two limits mean that: . Therefore, .

I checked the following source for guidance and used their notation: <http://www.mathamazement.com/Lessons/Pre-Calculus/11_Introduction-to-Calculus/limits-of-functions.html>

# Exercise 30

Justify each step of the derivation of the derivative of .

**Answer:**

In the first step, we define the difference , divided by .

In the second step we write the difference in terms of the given function.

In the third step we eliminate terms that cancel each other.

In the fourth step, we divide by .

Then, in the fifth and final step, we take the limit as approaches .

# Exercise 31

Find the first derivative of the following functions.

1. . []

# Exercise 32

Find the marginal for the given total.

# Exercise 33

Find the first derivatives of the following functions.

3. .

# Exercise 34

Find the first derivatives of the following functions.

# Exercise 35

Show that is on the line

**Answer:**

|  |  |
| --- | --- |
|  | In the above equation, set and the answer follows [. |

# Exercise 36

Find the differential for all functions in Exercises 31, 33, and 34.

*Exercise 31:*

. []

*Exercise 33:*

.

*Exercise 34:*

# Exercise 37

Suppose that . Find and approximate the value of . How does this answer differ from what we just found? Why would these approximations be different?

**Answer:**

. .

.

Compared to the actual value of (which is ), this is better approximation.

The reason why it is a better approximation is that the absolute value of in this case is smaller, namely versus in the example in the book.

# Exercise 38

Translate “ into English.

**Answer:**

There exists a positive number s.t. the value of the function is at least as large as the value of this function for any within the distance from .

# Exercise 39

|  |  |
| --- | --- |
| 1. Give a graphical example of a continuous function whose domain is not closed, so that it does not attain its maximum. | 1. Give a graphical example of a discontinuous function whose domain is closed and bounded but does not attain its maximum. |
|  |  |

# Exercise 40

Prove: Let be a real-valued function on . If has a local minimum at , and it exists, then

**Answer:**

We follow the prove of Theorem V.27 in the book (p. 109), except that we are looking at the minimum rather than the maximum of

Choose Choose . Then, because and because is a minimum. Then .

Choose . Then, because and because is a minimum. Then .

But and must be the same because is differentiable and the limit exists; they cannot assume two different values. Therefore, it must be true that .

# Exercise 41 (Solved using MS Math Solver)

Find the values of so that for each of the following functions.

1. . Solve for Solutions:
2. . Solve for Solution:
3. . Solve for . Solutions:
4. Solve for . Solution:
5. . Solve for . Solution:
6. Solve for . Solution:

Plots of Graphs (using <https://www.graphsketch.com/>):

|  |  |
| --- | --- |
| a. & b. | c. & d. |
|  |  |
| e. & f. |  |

# Exercise 42

Find the second derivatives for all the functions in Exercise 31, 33, and 34

Exercise 31:

1. ;
2. ;
3. ;
4. ;
5. ;
6. ;
7. ;
8. ;
9. ;

Exercise 33:

1. ;
2. ;
3. :
4. ;
5. ;

Exercise 34:

1. ;

# Exercise 43

Find the first and second derivatives for the following functions.

1. [second derivative obtained from <https://www.derivative-calculator.net/>]
2. ; ; [second derivative obtained from <https://www.derivative-calculator.net/>]
3. ; ;
4. ;
5. [ obtained from <https://www.derivative-calculator.net/>]
6. ;
7. ; ;
8. ;
9. ;
10. ; ;
11. for ;

# Exercise 44

Prove case (ii) of Theorem V.29: If then is a local minimum.

**Answer:** We follow the approach used to prove case (i) (see pp. 112-113)

If then . Since by hypothesis, we conclude that . Therefore, if , then and if , then . This mean that for is rising, and for is falling. Thus, in and is a local minimum.

# Exercise 45

For the functions in Exercise 41, determine which are maxima, which are minima, and which are neither. [Note: The graph in Exercise 41 also answer this question for each of the following functions)

1. . Solve for Solutions: ; is a local maximum. is a local minimum.
2. . Solve for Solution: . . is neither a local minimum nor a local maximum.
3. . Solve for . Solutions: ; ; , is a local maximum. , is a local minimum. , is a local minimum.
4. Solve for . Solution: . is neither a local minimum nor a local maximum.
5. . Solve for . Solution: . is a local minimum. is a local maximum.
6. Solve for . Solution: . . is neither a local minimum nor a local maximum.

# Exercise 46

Suppose that we are given the following total revenue and total cost functions. Use the above methods to find the output that maximizes profit where profit is total revenue minus total cost. When there are two critical values, how will you decide between them? Graph and marginal revenue and marginal cost.

|  |  |
| --- | --- |
| ; ; find such that : ; Profit is maximized when  A quick calculation shows that at , which is the smallest loss possible. |  |

In all three graphs ––– represents marginal revenue and ––– represents marginal cost .

|  |  |
| --- | --- |
| ; ; find such that : ; Profit is maximized when  A quick calculation shows that at , . |  |

|  |  |
| --- | --- |
| ; ; find such that : ; Profit is maximized when  A quick calculation shows that at , . |  |

In parts a, b, and c the maximum (minimum) occurs when . This can be seen in the graphs that goes with each part.

**Section VI: Calculus of Several Variables**

# Exercise 1

What is ? Write out the components of this sum. Now for , write out what equals and what equals. Solve the expression for and plug that into the expression. Now explain why the resulting equation is that of a straight line. Look back in the previous section where this matter is explained.

**Answer:**

.

;

; . The equation for is linear in , which means that its graph is a straight line.

# Exercise 2

For , treat as a constant and find the derivative with respect to . What do you get? How does your answer compare with the computation above (in the book)?

**Answer:**

. This answer is the same as in the book (“above”).

# Exercise 3

Find all the first partial derivatives of the following functions.

1. ;
2. ;
3. ; ;
4. ;
5. ; ;

# Exercise 4

Find the partial derivative of with respect to .

**Answer:**

# Exercise 5

1. Write the differential of
2. Write the differential of where and are constants:

1. For , find :

# Exercise 6

Find the differential to all functions in Exercise 3.

1. ;
2. ;
3. ;
4. ;
5. ;

# Exercise 7

Prove that if is a differentiable function and that has a minimum at , then .

**Proof:**

We follow that same approach as used in the book to prove Theorem VI.8 for the maximum.

Define , where and is differentiable because is differentiable (by assumption). Also: is a minimum because is a minimum (by assumption). Thus, by the theorem we proved for one variable (Section V).

. We can also write this as:

. When

Note that . In addition, is true for any . Therefore, .

# Exercise 8

For the following functions, find the critical values.

1. .

Solve →. Substitute into . ; (found using Excel and “trial and error”).

1. . .
2. . From the symmetry of the two derivatives follows that . Thus:
3. . From the symmetry of the two derivatives follows that . Thus: .
4. . ; ; I cannot solve this.
5. . ;

.

1. ; .

# Exercise 9

Find all partial derivatives for functions in Exercise 3.

1. ;
2. ; ;
3. ; ;
4. ; ;
5. ; ;
6. ;
7. ; ;

# Exercise 10

For each of the following functions, find the first partial derivative and the second partial derivatives (including all the cross partials).

1. ; ; ;
2. ;
3. ; ;
5. +360;

1. ; ; ;

2. ; ; ;
3. ;



;

1. ;

; skipped Below:



1. ; ;

;

;

;

# Exercise 11

Compute for and .

# Exercise 12

Compute for when there are only two variables where is

**Answer:**

First Step:

Second Step:

# Exercise 13

Find for all functions in Exercise 3.

1. ;
2. ;
3. ;
4. ; ;

1. ; ;

1. ;

1. ; ;

# Exercise 14

Write out the matrix product at the end of the remark of page 136 in the book and check to see that it is the same as our expression for that we got above. What are the dimensions of ?

**Answer:**

This is the same expression.

The dimension of is .

# Exercise 15

Compute for the following functions. Express this term as a matrix product.

1. ;

# Exercise 16

Find for the following functions and evaluate each at the given point. Express each in matrix form.

# Exercise 17

Write in matrix notation.

**Answer:**

# Exercise 18

In the above derivation (J. Hoag, pp. 137-140), we assumed that What if these terms are not equal; will the above result still hold? Here is a hint how to proceed. Assume and define . Replace in the computation above with .

1. Show that .
2. Proceed through completing the square on the term given in part a. Do you get the same result as we obtained above?

**Answer:**

We show that Define . This shows the correctness of part a.

We now complete the square: . Since at requires that and (since ); .

This is not the same result as when we assumed that .

# Exercise 19

Which functions from Exercise 16 have at the points given? How do you know?

**Answer:**

Only part c has We know because all first-order principal minors are negative and the second-order principal minor is positive. In a in b, two first-order principal minors are zero. In part, the non-zero first-order minors are positive. In part d, not all first-order principal minors are negative.

# Exercise 20

For the functions in Exercise 8, find and determine which critical values are maxima and which are minima.

1. .

Solve →. Substitute into . ; (found using Excel and “trial and error”; the exact solution is and using an online equation solver).

. By Theorem VI.14 (p. 138), the critical values are maxima because the first-order principal minors are negative and the second-order principal minor is positive (regardless of value of since . Both critical values are maxima.

1. . .

; because ;

1. . From the symmetry of the two derivatives follows that . Thus:

; because ; ; is neither a maximum nor a minimum.

1. . From the symmetry of the two derivatives follows that . Thus: .

; ; because ; ; is neither a maximum nor a minimum

1. . ; ; I cannot solve this and neither can the online equation solver.
2. . ;

.

; ; neither a maximum nor a minimum

1. ; .

; ; is a minimum.

# Exercise 21

Suppose that demand takes the following form. Find average revenue (AR) in each case.

# Exercise 22

Suppose we have that and . What values of and de we get? What if and .

**Answer:**

and : ; because .

and ;

# Exercise 23

Suppose that we have the following set of linear equations in variables. The are variables and the are the range variables. Does this system have an inverse? Under what conditions? Hint: Look back at the materials in Section III. Write the inverse of the system.

Rewrite in matrix form: . The inverse of exists if .

# Exercise 24

For each of the following, check to see if the condition in the Inverse Function Theorem requiring that the determinant is not 0 is satisfied. Are there points where this condition would not be satisfied? Then solve for and in terms of and .

1. condition is not satisfied for .

;

I do not know how to solve for and .

From second equation: . Use this result in the next step.

# Exercise 25

Check that our expression for is correct. Does this expression simplify?

**Answer:**

. ;

. It is correct and it can be simplified.

# Exercise 26

For each of the following find .

# Exercise 27

1. Consider . This is a function of three variables so and [comment: the text states , but this is a mistake]. Suppose we wish to solve for the variable in terms of and . To do this, we must be sure that the partial derivative of with respect to is not at the point whem.re we are trying to “invert” the system. Suppose we are trying to solve for in a neighborhood ofWhat will happen? Why? What is the problem here?

**Answer:**

. At , which violates one of the conditions under which the Implicit Functi8n Theorem (IFT) holds. Also, the determinant “collapses” to .

1. Suppose we have the following two equations in three unknowns Under what conditions can we solve for two of the variables, and , in terms of the remaining variable, , around the point .

**Answer:**

. . . is the condition that must be met. It is met for the point . Also,

. At the point

. At the point

1. Solve the following equation for in terms of and at the point and . Does the derivative condition hold? Can we solve for any one of the variables in terms of the other two? How do you know?

**Answer:**

. . . The derivative condition is met for all three variables for the point and .

. For For

. For For .

. For For .

1. For the following two equations, how many variables can we solve when , and ? How do you know? Solve for and in terms of .

Since we have two equations, at most we can express two variables in terms of the remaining variable.

**Answer:**

. .

. For there is no solution.

. For there is no solution.

. For there is no solution.

Either I have done this incorrectly or there is no solution because for the denominator is .

1. For the following three equations, how many variables can we solve for when and How do you know? Solve for and in terms of and

**Answer:**

Since we have three functions, at most we can solve for three variables in terms of the other variables.

for

1. For the following two equations, find the value of where the implicit function theorem does not hold when solving for any two variables in terms of the remaining two variables.

**Answer:**

First, both functions are . Second, the values of the variables must satisfy the two equations. One set of values that satisfies them is .

With two equations, we may be able to express two variables in terms of the other two variables.

.

The first determinant is zero if ; this occurs for and both of which satisfy the equations

The second determinant is zero if ; this occurs for

The third determinant is zero if ; this will not occur for rational numbers

The forth determinant is zero if

The fifth determinant is zero if

The sixth determinant is zero if

1. Suppose that we have a utility function . If we hold utility constant, we will be moving along an indifference curve. Thus, we would have , where is a constant. In this case we have and . We can use the Implicit Function Theorem to solve for in terms of and , which would be the indifference curve. However, if we do not know , we cannot write out the explicit equation for the indifference curve. However, we can use the Implicit Function Theorem to find thee slope of the indifference curve. What is the slope of the indifference curve?

**Answer:**

. The condition must be met.

1. Suppose that we have two linear equations in two unknowns. Under what conditions can we solve for two of the variables, say and , in terms of the other variable, ?

**Answer:**

First, we need values that satisfy both equations. Second, the two equations also must be continuously differentiable (which they are). Third,

1. Suppose we have the following two equations with the unknowns and , and and are constants.

Under what conditions can you solve for and in terms of and and some numbers? Solve these equations.

**Answer:**

This expression is not zero if and , and and , and .

;

We skip the calculation of , and .

1. In the statement of the implicit function theorem, we found an expression for . You can now see how this expression was obtained by following these steps.
2. Take the total derivative of each and set to .
3. Move the terms associated with to the right side of the equal sign.
4. Write the left side in matrix form as an matrix of partials of the with respect to times an matrix of the associated . On the right-hand side, write the system as an matrix of partials of the (with respect to ) times the matrix of

**Answer:**

1. Solve for the vector of on the left-hand side. What do you have to assume to do this?

**Answer:** must exist, or .

1. Solve for .

**Answer:** Apply Cramer’s rule.

1. Now allow all except for and solve for .

**Answer:**

# Note on Example on p. 161

Is convex? To be true, we must have ; .

Denote and . Let’s work on first.

.

Calculate . We can simplify this:

.

Divide by . Note that and therefore dividing by this term did not change the inequality sign.

Therefore or : is convex.

# Exercise 30

Restate the conclusion of Theorem VI.25 in terms of the principle minors of the matrix of the second partials of .

**Answer:**

We can write . The matrix is called the Hessian matrix.

A function is strictly convex if all principal minors of its Hessian are positive. It is convex it they are non-negative.

A function is strictly concave if the kth principal minors of its Hessian have the sign . It is concave it they are non-positive.

# Exercise 31

Suppose in Theorem VI.25 (that is, that the function is function of one variable). What does the proposition require in that case.

**Answer:**

Concave: . Strictly concave: .

Convex: . Strictly convex: .

# Exercise 32

Compute the second derivative of . What conclusion can you draw from this computation? Is convex? Strictly convex?

**Answer:**

. is strictly convex.

# Exercise 33

**Theorem VI.26:** Suppose that is a differentiable, real-valued function of variables defined on a convex set is a concave function if and only if

for every .

Restate this theorem for the case when .

**Answer:**

Suppose that is a differentiable, real-valued function of a variable defined on a convex set is a concave function if and only if

for every .

# Exercise 34

For a function of one variable, draw a graph of a concave function and show that the above condition is satisfied. What does this condition tell us? Draw a graph of a function that is concave on part of the domain and convex on the remainder of the domain. How does the above condition apply to the function you have drawn?

**Answer:**

The graph of is the tangent line to at . It tell us that no point of the graph of the function lies above this line (see graph on the left).

In the case of a graph of a function that is concave only in part of the domain, the theorem only to this part of the domain.

|  |  |
| --- | --- |
|  |  |

# Exercise 35

Use the first derivative characterization to show that is strictly convex. (HINT: Express as . Expand this term and use the first derivative characterization of strict convexity. See what is left over and complete the square on the remaining term.)

**Answer:**

Since

Hence, . is strictly convex.

# Exercise 36

Suppose that is convex. Restate Theorem VI.26 for this case.

**Answer:**

Suppose that is a differentiable, real-valued function of variables on a convex set is a convex function if and only if

.

# Exercise 37

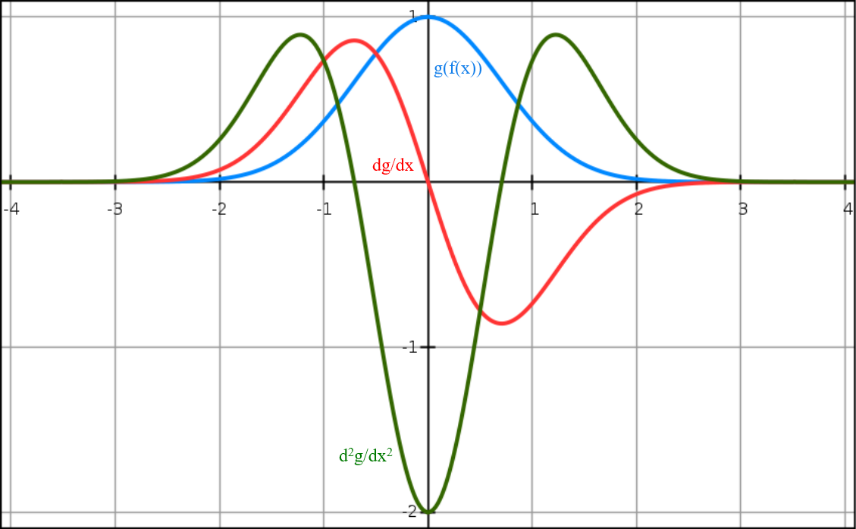
Convince yourself that is concave by computing the second derivative of . What do you get? How do you know is concave as a result of that computation? Now suppose we look at where . So . Compute the second derivative of . What sign is the second derivative? Is it always of one sign? Will be concave too?

**Answer:**

Part 1: . The second derivative is negative for all in the domain. Thus, is concave.

Part 2: . Since , the sign of the second derivative depends on . For . For . For . Finally, for

The graph of is symmetric since and reaches is maximum value of at The function is not concave (see graph of the function below) because we can find pairs of points on the graph that, when connected, lie above the graph of the function. An illustration showing the graphs of , , and follows.



# Exercise 38

Consider the function . What does this function look like? To help figure out what the three-dimensional graph must look like, do the following.

1. Graph the upper level set for

|  |  |
| --- | --- |
|  |  |
|  |  |

1. Graph the slice of the function with constant, say .
2. Graph the slice of the function with constant, say .

|  |  |
| --- | --- |
| Part b | Part c |
|  |  |

1. Graph the slice of the function with where is a constant.
2. Graph the slice along .

|  |  |
| --- | --- |
| Part d | Part e |
|  |  |

1. Is the function bounded? What does the graph look like? Is the function concave? Quasiconcave?

**Answer:**

The function is not bounded. The graph of the function forms a saddlepoint (see graph in Part a). The function is not concave. I do not think it is quasiconcave, either.

1. Use the determinant test to see if your conclusion is correct.

**Answer:**

Second order-preserving principal minor: : does not satisfy determinant test

First order-preserving principal minor: : satisfies determinant test

Conclusion: The function is not quasiconcave.

# Exercise 39

Suppose that . Is this function concave? Quasiconcave? What about . Is this function concave? Quasiconcave?

**Answer:**

.

.

Principal minors of Hessian:

This function is concave. If it is concave, it is also quasiconcave.

.

.

Principal minors of Hessian:

This function is not concave.

This function is not quasiconcave.

|  |  |
| --- | --- |
|  |  |
|  |  |

# Exercise 40

Work out the details for the function . Does this function have a minimum? Maximum? Is it concave? Convex? Quasiconcave?

**Answer:**

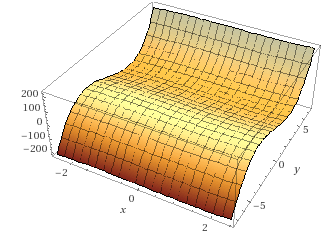
; We have four critical points, namely ,

; the sign is indeterminate

; the sign is indeterminate

Conclusion: The function is not concave nor quasiconcave and neither is it convex.

Graph of :



# Exercise 41

Which of the following functions is concave? Strictly concave? Convex? Quasiconcave? Quasiconvex? Assume that and .

Hessian: .

Hessian: .

Hessian: .

Hessian: . The sign depends on and on

It is clear that for , the Hessian. For , for large enough, . Finally, for , for large enough, .

Hessian: . Since the matrix is strictly positive definite, the function is strictly convex.

Hessian: . The function is strictly concave.

Hessian: .

Has indeterminate sign.

This function is concave, but not strictly concave (matrix is negative semidefinite)

Hessian: ; and

The matrix is neither negative nor positive semidefinite.

;

. This function is quasi-concave.

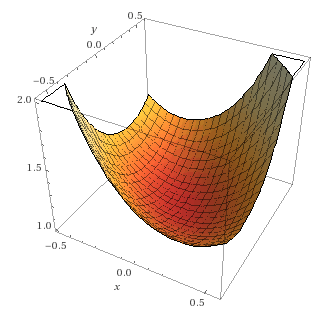
Hessian: ; and The matrix is neither negative nor positive semidefinite.

. Has indeterminate sign. The matrix is neither negative nor positive semidefinite

Hessian: . The sign of this determinant depends on . If then the matrix is negative definite in the interior of the domain and the function is strictly concave (in the interior of the domain).

Hessian:

Plot of this function:



# Exercise 42

Suppose and are concave, then is also concave. Does the result extend to quasi-concave functions?

**Answer:**

The claim that if and are concave, then is also concave, is obviously true. Since

and , then

.e c

Now we check if this also holds for quasi-concavity.

quasiconcave means that if and then .

quasiconcave means that if and then .

Then, on the convex set K: and and (also from the above inequalities) Therefore, the result extends to quasi-concave functions.

# Exercise 43

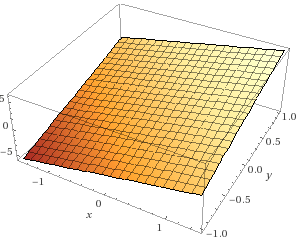
Suppose that where and are constants. Is this function concave? Convex? Quasiconcave? Note that the second derivative test fails.

**Answer:**

. Therefore, this function is convex as well as concave.

One of the properties of concave functions is that they are also quasiconcave.

Graph of the function for ad :



**Section VII: Techniques of Optimization**

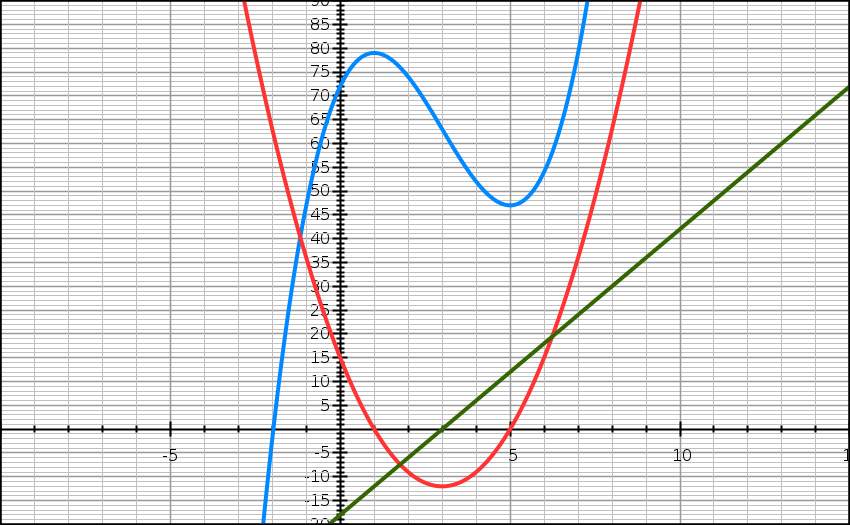
# Exercise 1

Find the critical values for the following functions. Which are maxima? Which are minima?

;

Conclusion: is a maximum; is a minimum.

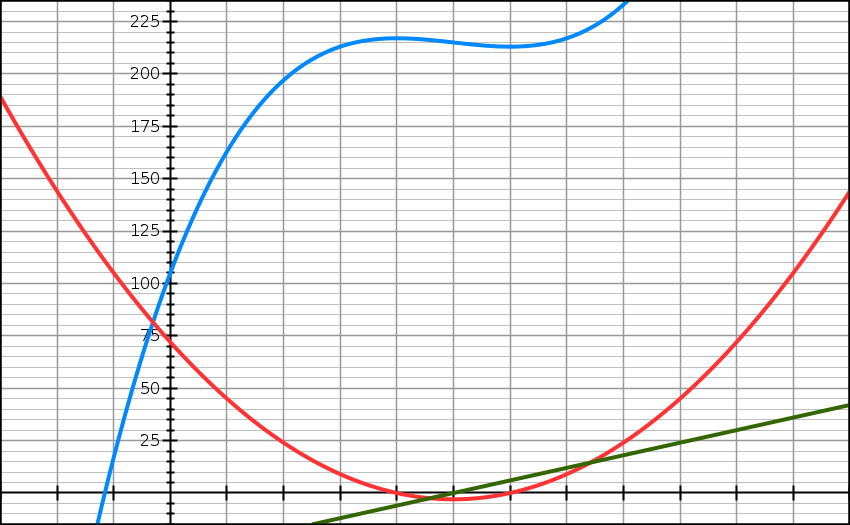
Graph of the function:

.

;

Conclusion: is a maximum; is a minimum.

Graph of the function:

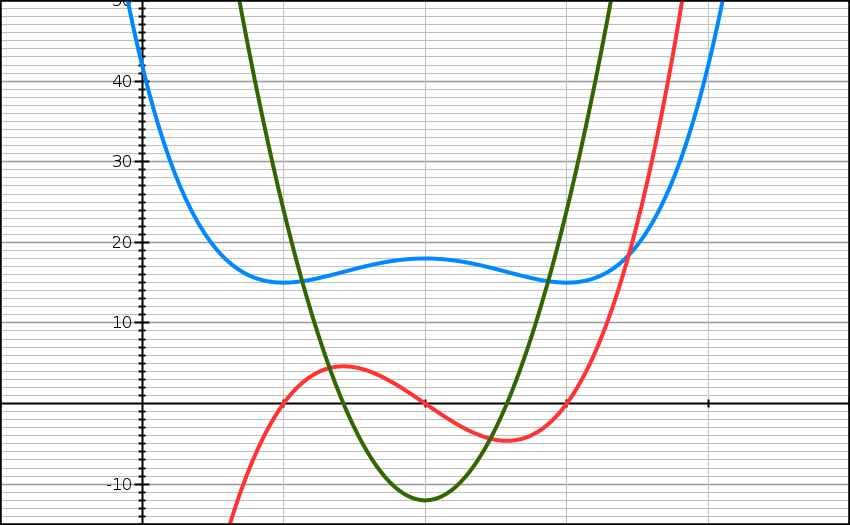


rewrite:

(obtained by inspection of graph. We could also have found it using the rational root theorem);

Conclusion: and are minima; is a maximum.

Graph of the function:



;

Conclusion: is a maximum; is a minimum.

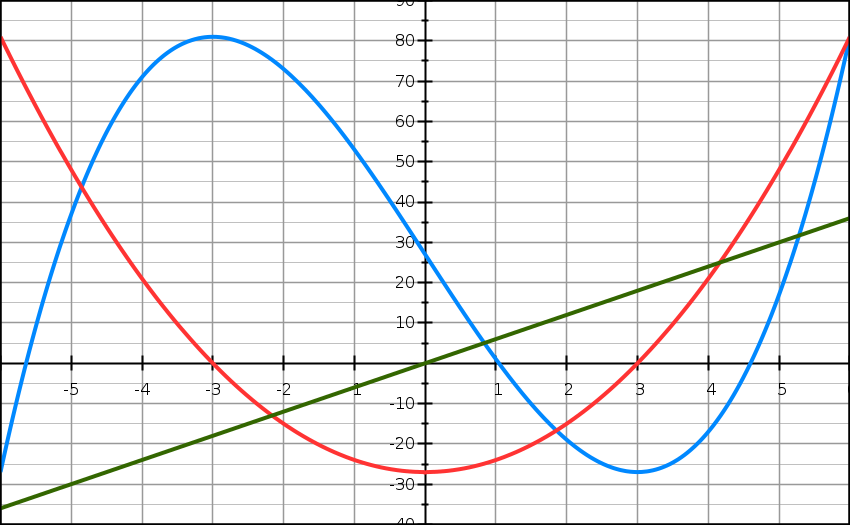
Graph of the function:



;

Conclusion: is a maximum; is a minimum.

Graph of the function:



# Exercise 2

Suppose that a firm has total revenue and total cost , where is output and is the tax rate. The tax rate is given.

1. Write the profit function. **Answer:**
2. Employ the first-order-necessary conditions to find the profit maximizing output.

**Answer:** (point B in illustration)

1. Employ the second-order conditions to see what outputs are maxima and which are minima.

**Answer:** . Change in total revenue should be less than the change in total cost at the profit-maximizing quantity . Because the tax rate is constant/given, it has no effect on the profit-maximizing output. The point C and the associated output minimizes profit and .

|  |  |
| --- | --- |
| Illustration: |  |

# Exercise 3

Find the points at which the following functions satisfy the first-order necessary conditions. Which are maxima? Minima?

.

; 0

is a minimum

. There is no solution. cannot equal **and** .

; .

and and . is a maximum.

; . is a maximum.

1. . This is the same as Exercise 3.a (see above).
2. . This is the same as Exercise 3.b (see above).

;

is neither a minimum nor a maximum.

;

. is a maximum.

;

;

. is a maximum.

It follows that there is no such that and . We cannot find a maximum or a minimum of

# Exercise 4

In each problem, solve the constraint for and plug this equation into the equation. Then find the maximum of this function.

**Answer:**

|  |  |
| --- | --- |
| Graph of |  |

**Answer:**

|  |  |
| --- | --- |
| Graph of |  |

**Answer:**

for . The second derivative is positive for .

|  |  |
| --- | --- |
| Graph of |  |

# Exercise 5

Find the critical values for the following constrained maximum/minimum problems.

. is neither a maximum nor a minimum.

is a maximum. is neither a maximum nor a minimum.

1. [note: the constraint is a circle with center at the origin and radius ]

From use this in ; use this in ; . The four possible solutions are

for for

. and are minima and and are maxima.

1. where and are positive constants. .

. .

.

. The sign of this determinant depends on the size of and . The important values of and are and Then the determinant is positive. If the determinant is negative. Both negative will not satisfy the constraint. Only one of them being negative cannot be a maximum. Therefore, both must be positive.

If and , the solution that meets of FONC is a maximum.

1. .

. .

. This has no solution where both and are non-zero.

From inspection we can derive the solutions. is the maximum and is the minimum. These solutions satisfy the FONC.

# Exercise 6

Suppose that we maximize subject to the constraints and . What is the solution? What problem do you encounter in this problem?

**Answer:**

These are the only solutions that satisfy both constraints. .

The constraints determine the solution; there is no “maximization” in a meaningful sense.

|  |  |
| --- | --- |
| Graph of Constraints: |  |

# Exercise 7

Maximize subject to the constraint Find the slope of the level curve of . At the maximum, what is the value of the slope? What is the slope of the constraint? How is the level curve related to the slope of the constraint at the maximum?

**Answer:**

.

; ; substitute this into the constraint:

. . The slope of :

. Slope at . The slope is the negative ratio of the coefficient of and , respectively, in the constraint.

|  |  |
| --- | --- |
| Graph of subject to |  |

Note the similarity of optimizing a utility function subject to an income constraint.

# Exercise 8

Do the computations (in the example starting on page 193) for the case where and and show that this condition cannot hold.

**Answer:**

The first two FONC then are:

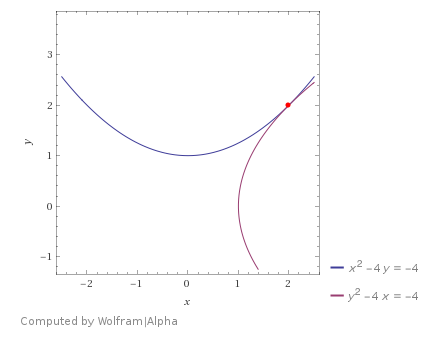
FONC becomes This is not possible. Hence, and cannot be true.

|  |  |  |
| --- | --- | --- |
| Contour lines of ( has been omitted because it does no influence the optimal choice of and of ) | Contour lines of constraint | Contour lines of constraint |
|  |  |  |

# Exercise 9

Graph the two constraints (in the example starting on page 193), and . What do you get?

**Answer:** There is only one point satisfying both equality constraints, namely .



# Exercise 10

How do we know that if there is a maximum, and ? Look at the graph with on the horizontal and on the vertical axis. We may ignore (why?) What do the level curves of the objective function look like? Where on the graph is the objective function maximum?

**Answer:**

|  |  |
| --- | --- |
| Objective Function. We can ignore because it can only assume the value in the optimal solution (maximum). plays no role in the constraints. | Contour lines of objective function. Blue shows “negative” contour lines (level lines). |
|  |  |

and yields the same value in the objective function as and . Both sets satisfy the inequality constraints and . In fact, and and and ( is an arbitrary number) satisfy the inequality constrains and, for yield a larger maximum than and or and . Therefore, the inequality constraints are not constraining how large and can become. Without the constraint, the objective function has no limit and there is no solution.

# Exercise 11

Graph the constraint in the above example (example on page 196). Also show the level curves for the function What do these level curves look like? Where is the maximum? Is the constraint effective?

Note: The objective function is . The single constraint is The equality constraint is a circle with center at the origin.

Answer on next page.

**Answer:**

|  |  |
| --- | --- |
| Level curves of | Constraint  The black dot shows the two the optimal solutions |
|  |  |
|  |

# Exercise 12

In the above example we found and as possible solutions. We then chose only and Show that is also a maximum but that and are not. Be sure to examine in the last two cases.

**Answer:**

and . Hence, if is a solution, then so is .

Although both and satisfy the constraint and , . Therefore, they cannot be maxima (also see graph in Exercise 11).

and .

From and : . This means that we have the following solutions: . Note that in solutions and , , which violates one of the FONCs. Thus, and cannot be solutions.

# Exercise 13

Suppose we have the same problem as given above except that must hold. Do this problem. What is the solution? How does the result differ from the above case?

**Answer:**

We have the objective function subject to and .

satisfies the constraints and .

If , then one of the variables must be zero. But then , which cannot be a maximum. Thus,

* Proceed with the assumption that . Then . From the first two FONCs: . The constraint can be written as ; then and . . . .
* Now proceed with the assumption that . From FONC 1 and 3: . Then, . violates the second constraint and cannot be a solution.

# Exercise 14

Where did we get the matrix in the last remark? How do we know that the rank is at least 1?

**Answer:**

The problem has two constraints: and . The objective function is . Then, the Lagrangean is .

and

The derivatives are:

This yields the matrix: . The rank is at least 1, because there is no row or column consisting only of zeros. In fact, the rank is 2 as can be seen if we take one of the first two columns and the third column and calculate the determinant, which is not zero.

**Section VIII: Odds and Ends**

# Exercise 1

Suppose we maximize subject to .

1. What function is playing the role of ? **Answer:**
2. What function is playing the role of ? **Answer:**
3. What is playing the role of ? **Answer:**
4. Write the Lagrange and find the that maximizes subject to .

**Answer:**

Maximize: . . .

1. What value does have at the maximum? **Answer:** .
2. Now write out the dual problem and solve it. What value do you use for ?

**Answer:**

Minimize: . . (from constraint); .

1. Do you get the same solutions for both problems? What about the multipliers? How are the multipliers for the two solutions related?

**Answer:**

Yes, we get the same solution: .

The multipliers are not the same. is the increase in of marginal increase in . is the increase in from a marginal increase in . .

# Exercise 2

Show the detail of the proof (of Euler’s Theorem on homogeneous functions) outlined at the top of page 204.

**Answer:**

Use the identity which follows from the definition of homogeneity of degree . Take the derivative with respect to on both sides. This yields the following equation: Since this is true for any , it is true for Therefore, if is homogenous of degree , then

, which is Euler’s Theorem on homogeneous functions.

# Exercise 3

Suppose we have two inputs, labor, , and capital, . The production function is .

1. Is the production function homogeneous? Of what degree?

**Answer:**

. The function is homogenous of degree 1.

1. Suppose we divide both sides by What do we get? Can we express this function in terms of ? How?

**Answer:**

. The output per worker is a function of the capital-labor ratio.

1. Show that is homogenous of degree .

**Answer:**

. Therefore, . This proves that is homogenous of degree .

# Exercise 4

Which of the following functions are homogenous? Of what degree? For those functions that are homogenous, show that Euler’s Theorem holds.

1. , where and are positive constants.

**Answer:** . Homogenous of degree .

.

**Answer:** . Homogenous of degree . . Therefore,

**Answer:** Homogenous of degree .

.

1. Suppose What is the degree of homogeneity of ?

**Answer:** . If then . If is homogenous of degree , then is homogenous of degree and so is . Also, if is homogenous of degree , then . Hence . Conclusion: is homogenous of degree .

# Exercise 5

Suppose that the demand for good is homogenous of degree . Consider now the function . Of what degree is homogenous? Find the derivative of with respect to . Of what degree is this function homogenous? Show your work!

**Answer:**

. We made us of the fact that is homogenous of degree , that is, is homogenous of degree .

. is homogenous of degree (by assumption). Therefore, is homogenous of degree by Theorem VIII.4. is homogenous of degree . Therefore, is homogenous of degree . Conclusion: is homogenous of degree . This is as it should be according to Theorem VIII.4.

# Exercise 6

Show that is not homogenous. (HINT: use Euler’s Theorem)

**Answer:**

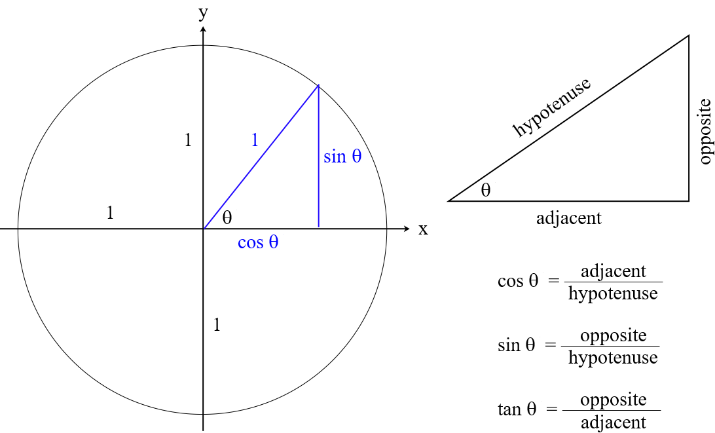
Let . . Then . Therefore, is not homogenous.

# Exercise 7

Use the previous remark to prove the proposition just stated.

Theorem: Suppose and are two vectors from Then and are perpendicular (orthogonal) if and only if

**Answer:**

From trigonometry we know that . We also know that for See graph.

**Section IX: Preferences, Utility, and Demand**

# Exercise 1

**Background:** Consider . Every element of has the form . Consider two elements from and . Define the complete preordering as follows. We say if and with one of the inequalities strict. This is a Pareto ordering.

**Exercise:** Show that Pareto satisfies transitivity.

**Answer:**

Assume and . Then , because and or and or and . Since we have and with one of the inequalities strict. Thus, we can write and or and or and . Hence, and with one of the inequalities strict. Conclusion: Pareto satisfies transitivity.

# Exercise 2

**Background:** Lexographic larger. Consider and . We define as follows. We say if or if and . We say if or if and .

**Exercise:** First show that is complete and transitive, but not reflexive. Second, draw and locate a point in . Find all so that . Find all so that .

**Answer:**

Chose an arbitrary and . Either (1) or (2) or (3) . In this third case we have either (1) or (2) or (3) . But if and , then . Therefore, for , is complete.

Assume that and . Then, we have if or and . Therefore, and imply and is transitive.

is impossible because and are impossible.

|  |  |
| --- | --- |
| **Graph:**  The line that separates the yellow from the reddish area is part of the region where above the dotted line.  The line that separates the yellow from the reddish area below the dotted line is not part of the region where . |  |

# Exercise 3

Show that the length is complete and transitive.

**Answer:**

We say that is . We can immediately see that length is not reflexive because is not possible.

If , then . This shows that distance is transitive.

I do not know how to show completeness because it is not possible that or or both. It is the “or both” part that defeats me.

# Exercise 4

Suppose . Show that cannot happen.

**Answer:**

The proof follows from Theorem IX.II.

# Exercise 5

Suppose . Show that .

**Answer:**

If , then either or . But this contradicts . Therefore, .

# Exercise 6

Suppose that and are defined as above. Show that we cannot have and .

**Answer:**

This follows from the answer to Exercise 5 that implies .

# Exercise 7

How might we use [as good as] and [not better than] to find ?

**Answer:**

If a bundle, , is and , then and lies on the indifference curve through .

# Exercise 8

Suppose that there are two goods. Draw the axis system for these two goods. Choose a specific commodity bundle, Find the set using lexicographic preferences. Also find the set for the same preferences.

|  |  |
| --- | --- |
| **Answer:** |  |

# Exercise 9

For lexicographic preferences, what is the indifference curve for the bundle ? Is this indifference curve continuous?

**Answer:**

Examining the graph in Exercise 8 shows that combinations of in the red quadrant (which includes the border above but not below ) are “better” because they either offer more of and at least as much of , or the same amount of and more of . By similar reason it can be shown that points in the yellow area are less preferred. Only the initial allocation, shown as a dark dot in the graph, is the indifference “curve.” (A point is “trivially” continuous. We can use the “neighborhood” definition, which has only one element and is, therefore, “trivially” met.). The author, John Hoag states that a point function is not continuous, which is true if we use the definition in terms of limits.

# Exercise 10

Look at the set for lexicographic preferences. Show that these preferences are not continuous.

**Answer:**

We know that for all . But the limit . Hence, the limit is not included in the set of bundles . The preferences change suddenly (discontinuously).

Acknowledgment: <http://web.uvic.ca/~egugl/indexecon400_files/lecture1.pdf>

# Exercise 11

Are the lexicographic preferences convex? Explain why or why not.

**Answer:**

Assume , that is, or, if , then . Choose . Then because either or, if , then . Therefore, lexicographic preferences are convex. Using the same approach but strict inequalities, the answer can be extended to show that lexicographic preferences are strictly convex.

# Exercise 12

Are the lexicographic preferences monotone? Explain why or why not.

**Answer:**

Yes, lexicographic preferences are monotone because if then , or if and , then .

# Exercise 13

Suppose that the consumer’s ranking process only satisfies the monotone condition. That is, if then holds. Does this mean that the consumer has preferences? Does this condition alone satisfy both completeness and transitivity?

**Answer:**

It must be true that either or or both. Hence, it must also be true that either or or both. Therefore, monotonicity implies completeness.

Additionally, it must also be true that if and then . Then it is also true that if and , then , which shows transitivity.

# Exercise 14

In the two goods case, you should graph the budget constraint on an axis system with on the horizontal axis and on the vertical axis. What happens to the budget line when drops? What if changes? What is the slope of the budget line? The intercept?

|  |  |
| --- | --- |
| **Answer:** |  |

# Exercise 15

Show that if , then .

**Answer:**

. Then .

# Exercise 16

Solve the expression for to show that . What are and in this case?

**Answer:**

and .

# Exercise 17

In the discussion above, we started with a so that the consumer liked the same as Is this particular choice of important for the discussion? What if we start with a so that the consumer likes twice as much as ? What difference will this make in the discussion?

**Answer:**

|  |  |
| --- | --- |
| or  .  .  Now we can proceed as in Exercise 16. |  |

# Exercise 18

Suppose that a consumer has the utility function where and are positive constants and add to some positive number. This is the Cobb-Douglas function. Write the out the standard budget constraint and the first-order necessary conditions for a maximum.

**Answer:**

.

Discussion: It follows from the FONC that . In other words, to optimize utility we spend on and in proportions equal to over . If then we spend (expenditure) more onthan Note this does mean that we buy (quantity) more of than of ; that depends on the relative prices.

# Exercise 19

Suppose that we have the consumer’s problem, but now we wish to also be sure that each is greater than or equal to What would the Lagrange be now (use Kuhn-Tucker VII.16)? What would the first-order conditions be?

**Answer:**

The Lagrangean is the same in Exercise 18: , but the FONC have changed: .

What this means is that if not all the budget is used, then In Exercise 18 the formulation required that the whole budget must be used.

# Exercise 20

Find the total derivative of the utility function. Find the total derivative of the budget constraint. Assume that the variables include all the , all the , and .

**Answer:**

.

is the change in the budget if the price of changes. is the change in the budget if the quantity of changes.

# Comment on FONC (pp.232-233)

.

Then, FONC are:

and

From this follows: . Find . Let except for and Then we conclude that .

Use the Envelope Theorem to find: , which is the marginal value of more money in the budget.

# Exercise 21

Find the conditions that would have to hold so that the hypothesis of the Implicit Function Theorem would be satisfied so that we could solve the first-order necessary conditions for the demand curves.

**Answer:**

The bordered Hessian must exist and be invertible.

# Exercise 22

Use the first-order necessary conditions from Exercise 18 and find the demand curves for both and .

**Answer:**

From FONC; substitute for

Demand for : substitute into

Compare this result to the discussion in the answer to Exercise 18.

# Exercise 23

Of what degree is the homogeneity of demand? Suppose that all prices and income are multiplied by the same constant, . What happens to the quantity of any good the consumer would choose? How would you draw this impact on the demand curve?

**Answer:**

The term in the demand for and in the demand for show that multiplying all prices and income by the same factor will not change demand. This means that the demand curves are homogenous of degree .

|  |  |
| --- | --- |
|  | The blue is the “original” demand curve, the red the demand curve after income and price both have doubled. After the doubling of price and income consumer consumers offer twice as much as before for the same quantity demanded. The demand curve has shifted upward. |

# Exercise 24

Suppose that in the two-good case, a consumer has lexicographic preferences and a standard budget constraint. Draw the graph of the budget constraint and find the bundle that maximizes the consumer’s preferences in that budget set. What conclusion can you draw about demand? Write the equation of the demand curve.

**Answer:**

|  |  |
| --- | --- |
|  | The consumer will spend all income on the most preferred good.  The demand curve is , where income. |

# Exercise 25

Compute and set to . From Eq. F1[[1]](#footnote-1) find and substitute into . Then find the total differential of the budget constraint and use the equation to simplify the derivative of the budget constraint. Use the result in \* (on page 235) together with . The resulting term is the substitution effect, . What did you get for the substitution effect?

**Answer:**

.

Use from F1: .

From this follows: for .

The total differential of the budget constraint: .

Since the last term is .

Now use to obtain . Finally, assuming , the differential of the budget constraint is (the left-hand side shows how much the consumer saves – how much purchasing power she gains (loses) – if the price of good drops (raises) and the quantity consumed of that good stays the same).

# Application: The Two-Goods Case

|  |  |
| --- | --- |
| The FONC are: | The total derivatives of the FONC are: |

In the two-goods case the bordered Hessian is:

. is symmetric because .

If is quasi-concave and has full rank, then .

.

.

If and , then .

. The first is the income effect and the second part the substitution effect.

# Exercise 26

For the two-good case, graph the utility maximizing choice. Now reduce the price of holding the price of and income constant. Find the new utility maximizing choice. The difference between these two utility maximizing choice is the change in quantity demanded. Now, by geometrical means, find the income and substitution effects. Consult an intermediate level textbook if necessary.

**Answer:**

|  |  |
| --- | --- |
|  | If we hold utility constant, the substitution effect of a lower price for good increases the quantity demanded of that good to . The income effect increases it further from to .  .  is the total budget.  For the other good, the substitution effect of a lower price for good reduces the quantity demanded from to , and the income effect increases the quantity demanded from to . |

# Exercise 27

For the demand functions you found in Exercise 22, find the slope of the demand, the income effect, and the substitution effect without using the method of comparative statics. How can you find these effects without doing the total differential of the first-order conditions?

**Answer:**

Demand curves: ;

Slope of demand for and slope of demand for

The demand functions show that demand depends only on the budget and the good’s own price. Therefore, we can use budget constraint to obtain the income effect: from follows that the income effects of a price change are, respectively: and . Because demand does not depend on the price of the other good, there is no substitution effect.

is the amount allocated to purchasing . It depends only on the parameters . is the amount allocated to purchasing .

.

# Exercise 28

Suppose that a consumer has preferences between two goods that are perfect substitutes (what will the indifference curves look like? Can you use calculus to solve this problem?) Can you change prices such that the entire quantity demanded is due to the income effect?

**Answer:**

|  |  |
| --- | --- |
|  | The indifference curves are straight lines.  The red lines are budget lines. . Therefore, the absolute value of the slope of the line Budget 2 is less than that of line Budget 1. stays the same and the dollar amount of the budget also remains unchanged.  The blue lines are indifference curves. They are parallel to each other.  When the price of is , then **1** shows the optimal consumption: only good is purchased.  When the price of is , then **2** shows the optimal consumption: only good is purchased.  Calculus cannot be used because the choices do not vary continuously as the price changes. |

Can you change prices such that the entire quantity demanded is due to the income effect? No, there is either no effect or the effect is entirely due to substitution.

# Exercise 29

Show that it is not possible for all goods to be inferior.

**Answer:**

An inferior good is one whose consumption drops as income goes up. If all goods were inferior goods, then consumption of every good and total spending (since prices have not changed) would decrease. But this must mean that utility would also decrease, which violates the assumption of utility maximization. Alternatively, if consuming less of every good yield as higher utility, then the properties of a utility function are violated.

# Exercise 30

We know that demand is homogenous of degree 0 in prices and income (Exercise 23). For the case of two goods, and with prices and , and income , apply Euler’s Theorem to the demand for . Now divide the resulting expression by . What does this tell you about the elasticities of demand?

**Answer:**

Applying Euler’s Theorem to : . Dividing by yields:

price elasticity of cross-price elasticity of income elasticity of .

# Exercise 31

Suppose that the consumer has a utility function that is additively separable, that is, . Show that in the two-good case the income effect is positive.

**Answer:**

|  |  |
| --- | --- |
| The FONC are: | The total derivatives of the FONC are: |

In the two-goods case the bordered Hessian is:

. If is quasi-concave and has full rank, then .

.

. Set and and use Cramer’s Rule.

.

. The first term on the RHS is the income effect and the second part the substitution effect. The substitution effect is negative because , which is correct. The income effect is also negative, which I believe is wrong.

# Exercise 32

Suppose that a consumer has indifference curves that are vertically parallel (so utility might be , for example). Find the income effect in this case.

**Answer:**

|  |  |
| --- | --- |
| The FONC are: | The total derivatives of the FONC are: |

Set and .

. There is no income effect.

# Exercise 33

Suppose that the consumer has a utility function , where and are positive constants. Use the standard budget constraint and find the demand curves for this case. How are the demand curves different than those that you found in Exercise 22? How is this utility function different from that in Exercise 18?

**Answer:**

The FONC are:

Derive demand curves:

From FONC: ; .

; (from budget constraint)

.

. In the same way we obtain .

The demand functions are the same as those obtained in Exercise 22.

The utility function can be obtained by taking the natural log of the utility function in Exercise 18. This transformation preserves the rankings of “bundles” of in Exercise 18.

# Exercise 34

Now by the Envelope Theorem, we can find how changes with a change in . How? What do we get? We can also see how changes as changes. How? What do we get?

**Answer:**

Let be the demand for that maximizes utility subject to the budget constraint. Then, given prices and the budget, is the maximum utility attainable. Assuming the assumptions of the Envelope Theorem are satisfied (see page 207 in text), then because or it would not be an optimal (critical) value.

Similarly, .

# Exercise 35

Assume that a consumer has a Cobb-Douglas Utility function and two goods. Find the demand curves from utility maximization and find the indirect utility function. Now find the derivative of the indirect utility with respect to . Find the derivative of the indirect utility with respect to . Do you get the result suggested by the previous exercise?

**Answer:**

We solved the optimization problem in Exercise 22 and obtained and .

The indirect utility is: .

.

.

I am not sure how to answer the last part of the question, but I think the answer is yes.

# Exercise 36

Interpret the first-order conditions (in the dual consumer optimization problem). Is the interpretation different from the utility maximization case?

**Answer:**

From the FONC follows that or, alternatively: . This means that in the optimal solution the marginal utility per dollar spent is the same for both goods. The interpretation is the same as in the utility maximization case.

# Exercise 37

Explain why each equality in the following line is true.

**Answer:**

The equalities are true because of the assumption and and .

# Exercise 38

Suppose that the government is considering, in a two-good world, a sales tax on good 1 or an income tax. Suppose that the taxes are designed to raise the same amount of revenue. Show, using revealed preference, that the consumer would prefer the income tax to the sales tax.

**Answer:**

The initial consumption consists of the bundle and costs

With a sales tax, the consumption expenditure is .

With an income tax, the consumption expenditure is .

By assumption: . Therefore, .

With an income tax, the bundles and are both feasible (cost the same after paying sales, or income tax, respectively), but the bundle is chosen. Therefore, the bundle is revealed preferred. In other words, the consumer prefers the income tax to the sales tax.

because in the case of the income tax: , but in the case of the sales tax: .

|  |  |
| --- | --- |
|  | To the left is the graphic representation of the result. Note that the green (sales tax) indifference curve is tangent to the green budget line at the point where the green and the red (income tax) budget lines intersect. If this were not the case, then the sales tax and the income tax would not generate the same revenue.  Point A shows the optimum without a tax.  Point ITax shows the optimum with an income tax.  Point STax shows the optimum with a sales tax.  Between the two taxes, the income tax allows a higher level of utility than the sales tax.  Therefore, the income tax is preferred. |

# Exercise 39

Suppose that the consumer has lexicographic preferences. Show that the strong axiom is satisfied for these preferences. The strong axiom says that the consumer acts as if she maximizes utility subject to the budget. What would the utility function be in this case?

**Answer:**

Compare two commodity bundles and . if has more of the most preferred good than or the same, but more of the second most preferred good, or the same, but more of the third most preferred good, etc. Then, if it follows immediately that . We can show the same for all goods.

I am not sure how to answer the last question. Didn’t we state that the consumer has lexicographic preferences? Why would this change?

# Exercise 40

For the following utility functions, find the coefficient of absolute risk aversion.

1. **Answer:**
2. , where and are positive constants.

**Answer:**

1. . **Answer:**

# Exercise 41

For the following utility functions, find the coefficient of relative risk aversion. How does it differ from the coefficient of absolute risk aversion?

1. **Answer:**
2. , where and are positive constants.

**Answer:**

1. . **Answer:**

You obtain the coefficient of relative risk aversion by multiplying the coefficient of absolute risk aversion by

# Exercise 42

Apply the following transformation to the utility function . Then compute the coefficient of relative risk aversion. What has happened to the coefficient?

1. .

**Answer:**

.

.

**Answer:**

.

.

**Answer:**

.

# Exercise 43

Suppose that a consumer has the following utility function, , where is spendable income. Suppose that the consumer is faced with a choice between not buying insurance or buying insurance. Suppose that the consumer’s income is $50,000. The possible loss is $10,000 which occurs with probability 0.1. The insurance will fully pay for the loss.

1. Find the expected income if the consumer buys no insurance.

Answer:

.

1. Now assume that the premium is $1,000. What will the consumer’s spendable income be if there is a loss? What is the expected spendable income in this case?

**Answer:**

In case of loss

.

1. Find the utility of the consumer’s income.

**Answer:**

.

1. If the consumer does not buy insurance, she would have an income of 50,000 with probability 0.9. How much spendable income would the consumer have if there were a loss? What is the probability of that level of income? What would the expected utility be in the case of a loss?

**Answer:**

In case of a loss, the spendable income

The probability of this income is 0.1, which is the probability of a loss.

The expected utility in this case is .

1. Now suppose that the consumer buys the insurance and pays a premium of $1,000. How much spendable income would the consumer have if he/she bought the insurance and there were no loss? What would the spendable income be if he/she bought the insurance and there were a loss? What would the expected utility be now?

**Answer:**

Without loss: with loss:

.

1. Would the consumer be willing to buy the insurance at this premium? Explain! How much would the premium have to be before the consumer would not be willing to buy the insurance?

**Answer:**

. Consumer will buy insurance.

The breakeven point occurs when , the same as the expected utility without insurance. This expected utility corresponds to .

. This is the maximum premium the consumer is willing to pay.

**Section X: Supply**

# Exercise 1

1. Show that TC and TVC are parallel

**Answer:**

. Since by definition, its slope is 0. Therefore, and have the same slope (are parallel). Alternatively: . Since TFC is constant, and the result follows.

1. Show that ATC = AVC + AFC.

**Answer:**

.

1. Show that AVC and ATC are not parallel.

**Answer:** Slope of . . . Since the numerator of is not constant but is constant for , and they both have the same denominator, they cannot be the same.

1. Show that MC is also the slope of TVC.

**Answer:**

First, we already showed that the slopes of TC and TVC are the same. By definition, , which is the slope of TC and TVC, respectively.

1. Show that at the output where ATC has its minimum, ATC = MC.

**Answer:**

at the minimum. Multiply both sides by

.

1. Show that for a monopoly where , that .

**Answer:**

.

# Exercise 2

The minimization problem has two elements. The first is expenditure on variable inputs, which we minimize, and the second is output, temporarily held fixed, produced from those inputs. For now, we will maintain the hypothesis that the firm is competitive in both the sale of output and the purchase of inputs. Let be the price of output and be the price of the ith input.

Write an expression for the expenditure on variable inputs. This function is called the **isocost**.

**Answer:**

. The graph of this function is linear.

# Exercise 3

Suppose that one output is produced from two variable inputs, Find an expression for the slope of the isoquant (HINT: use the total differential if the Implicit Function Theorem).

**Answer:**

We use the total differential: for the isoquant (no change in output). Then, the slope is . In other words, the slope is the ratio of the marginal products of the two inputs.

# Exercise 4

Suppose that the production function is a Cobb-Douglas form, . What is the slope of this isoquant?

Answer:

and . Then, is the slope. The slope is negative.

# Exercise 5

We often draw the isoquant so that the upper level set is convex (the set of the input combinations producing at least is convex). For the general production function, find the conditions that would have to be satisfied for this convexity to hold (HINT: What would the slope of the isoquant have to do for the isoquant to display this convexity?).

**Answer:**

The marginal products must be positive for the isoquant to be convex: if we increase use of one input while keeping output fixed, we must decrease the use of at least one other input. What this means is that the slope of the isoquant must be negative.

# Exercise 6

The problem is to minimize expenditure on variable inputs to a restriction that constant, say . Write the Lagrange for this problem.

**Answer:**

Write the FONC for this minimization. Interpret the multiplier.

**Answer:**

|  |  |
| --- | --- |
|  | The multiplier times the marginal product must be equal to the unit cost of in the cost-minimizing solution.  . |

In the case of two inputs, draw a graph illustrating the minimization problem and the solution you have found. Explain how the FONC direct you to the point you say is minimum.

**Answer:**

Next page

|  |  |
| --- | --- |
|  | We know the slope of the isoquant (from Exercise 3): .  We also know the slope of the expenditure line: .  At the optimum we have or |

You can use the Implicit Function Theorem to solve for the ’s as a function of , and Under what conditions can you do this? The functions you are finding are the input demand curves conditional upon the output, called conditional input demand curves.

**Answer:**

and the expenditure function, , must be twice continuously differentiable. Also, the determinant

What is the sufficient condition for this minimum subject to the output constraint? How is this condition related to the condition due to convexity of the isoquant that you found in Exercise 5?

**Answer:**

We need to have the determinant (above) to be positive semidefinite . This is the same as requiring the isoquant to be convex.

Now carry out the comparative statics. What can you tell about terms like ? What about ?

**Answer:**

Next page

. In the case of two inputs, this becomes

. Setting results in

. Setting results in .

# Exercise 7

How is the problem you just solved similar to and different from the utility maximization problem that was solved earlier in the notes?

**Answer:**

Mathematically, they are very similar. In fact, if we take the negative of the objective function of the cost minimization problem, we are back to a maximization problem. The difference is the economic meaning and the properties of the objective functions (concave or quasi-concave vs. convex or quasi-convex).

# Exercise 8

Suppose that the production function is Cobb-Douglas with parameters adding up to strictly less than 1. Assume two inputs. Carry out the cost minimization problem and find the conditional demand for inputs in this case.

**Answer:**

. Use this in constraint: . It follows that

. Thus, .

Checking second-order conditions shows that we have a minimum:

because .

.

# Exercise 9

How can we find total variable cost from the information we have so far generated?

**Answer:**

# Exercise 10

Use the output from the Cobb-Douglas case (Exercise 8) to find the TVC.

**Answer:**

.

# Exercise 11

Find marginal cost in the Cobb-Douglas case (Exercise 10). What would happen if the parameters add to exactly one?

**Answer:**

. If , then constant.

# Exercise 12

Suppose that the firm has the production function , where K is capital and L is labor. Use the standard isocost and find the cost minimizing choice for the firm. Provide a graphic solution to this problem. Check the second-order conditions.

**Answer:**

. I replace “min.” with “opt.” because we need to check if we get a minimum or maximum.

. In other words, capital and labor are used in proportion to their unit costs. Use this to find . Then: . Similarly, we can obtain an expression for K.

Check of second-order conditions shows that we have a maximum (see graph on next page). The solution gives the highest possible cost for the output given by the isoquant.

.

Graphical solution:

|  |  |
| --- | --- |
|  | The mathematical solution above is a maximum, not a minimum. As drawn and indicated by the slope of the cost line, capital is cheaper than labor. Since both contribute equally to output, it makes always sense to only use one input, namely the cheaper one. Even if the cost is the same for both, it still makes sense to only use one input, although in this case it makes no difference which one. The reason is that, if , .  Calculus does not yield the correct solution because we have a corner solution and production function displays increasing returns to scale. |

**Comment 1:**

Let’s solve the dual problem instead.

As above, ; capital and labor are used in proportion to their unit costs. .

. We can find a similar expression for capital.

Check of second-order conditions shows that we have a minimum (see graph above), namely the smallest possible output for the budget given by the tangent line to the isoquant. Note that the isoquant is concave, not convex.

.

The graphical solution is the same as before, that is, a corner solution. The solution obtained with the Lagrangean would result in the most expensive way to produce a given output, the is, the minimum output for a given TVC.

**Comment 2:**

Let’s solve the following problem instead.

. We will check, but in this case we obtain a minimum.

Check second-order conditions:

.

We have minimum.

Graphical solution of :

|  |  |
| --- | --- |
|  | Notice that in this case the Langrangean approach gave us the correct solution because the optimal solution is not a corner solution.  Notice also that in this case the isoquant is convex, which is why we obtain a minimum.  The production function displays decreasing returns to scale so that, if , . |

For completeness sake, we also solve the dual to the problem in Comment 2:

Second-order conditions:

.

We have maximum.

# Exercise 13

Use the Envelope Theorem to find the slope of TVC.

**Answer:**

.

. The slope is positive, but decreasing with , which reflects increasing returns to scale.

# Exercise 14

Suppose that we have a single output, Q, produced from a single input, labor. Thus we have . Write an equation for the total variable cost in this case. The total cost.

**Answer:** ; .

Assume that output is a linear function of labor so more specifically, where is a constant. What is the equation for the total variable cost curve in this case?

**Answer:** .

What is the profit function in this case?

**Answer:**

Find the profit maximizing output.

**Answer:**

Write out the profit function and use the FONC to find the profit maximizing quantity of labor.

**Answer:** . Hence:

Illustrate your answer with a graph.

**Answer:**

|  |  |
| --- | --- |
|  | . |

# Exercise 15

What impact would you expect for each of the following on supply?

1. A 70% tax on profits.

**Answer:**

No effect.

1. A tax of $1 on each unit produced.

**Answer:**

Reduction in supply.

1. A license fee of $1,000 to operate the business.

**Answer:**

No impact, as long as the price is sufficient to cover the additional cost, which is a fixed cost.

as before, since .

1. A tax on what the business pays its workers.

**Answer:**

Reduction in supply because the price of the input labor has increased. A change in the cost per worker changes the marginal cost.

# Exercise 16

Reformulate the profit maximization problem to include a constraint that requires . What will the FONC for this be? What do the conditions tell you? How does this condition relate to the shutdown condition given above (Hoag pp. 261/2).

**Answer:**

We write the constraint as .

.

; .

If and we are back to . If

If and : the price must be higher than MC to start production.

# Exercise 17

Form a ratio of the first-order conditions (divide the first-order condition for by the first-order condition for with Compare this condition to the ratio you obtained in the cost minimization problem. What conclusion do you draw?

**Answer:**

.

.

The same condition holds in both cases (see Exercise 6). The ratio of input unit prices is equal to the ratio of the inputs’ marginal products.

# Exercise 18

How do you know that the slope of the input demand is negative?

**Answer:**

The total derivative of the first-order conditions in matrix form is:



.

.

, where is the cofactor of element in row , column. If we set

, we obtain because the sign of is the opposite of the sign of .

# Exercise 19

Why is How do you know?

**Answer:**

The expression for . The term in parentheses is a quadratic form and is the inverse of the Hessian, which in this case (maximum) is negative definit. The result follows.

# Exercise 20

Suppose that a firm has a Cobb-Douglas production function with two inputs and with the parameters adding to 1, adding to less than 1, adding to more than 1. Find the input demand and supply of output in each case. What conclusions can you draw?

**Answer:**

.

. . Substitute into production function: . Thus: .

demand for capital is proportional to output. .

demand for capital increases more than proportionally to output because . It is easy to see that

demand for capital increases less than proportionally to output because .

The result for labor is similar because .

. If , the output is any amount consumer demand at that price.

.

. .

# Exercise 21

Suppose that all prices are multiplied by the same positive constant, t. What will happen to the quantity produced by this firm?

**Answer:**

From the result in Exercise 20 it follows that such a change will leave the quantity supplied unchanged.

# Exercise 22

Under what conditions will we have ? ? Under what conditions can we have and both and ? What if

**Answer:**

and we use the suggestion by Hoag (p. 266).

First, let to find the slope of the isoquant: if both partial derivatives have the same sign; otherwise.

Second, let to find the slope of the transformation curve: ;

if both partial derivatives have the same sign; otherwise.

Third, let .

Fourth, let .

Finally, let .

The conditions asked about in the question can be deduced from these results.

# Exercise 23

Write out the first-order necessary conditions and interpret them. Then use the method of comparative statics to find slopes of input demand and the slope of output supply. What do you get?

**Answer:**

.

.

; ; .

We can use the following matrix for comparative static analyses.



|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | … |  |  | … |  |  |  |  |  |  |
|  | … |  |  | … |  |  |  |  |  |
|  | … |  |  | … |  |  |  |  |  |
|  | … |  |  | … |  |  |  |  |  |
|  | … |  |  | … |  |  |  |  |  |
|  | … |  |  | … |  |  |  |  |  |
|  | … |  |  | … |  |  |  |  |  |

If we have only one input and one output, then , as expected. , also as expected. in this case with (see Hoag, p. 187).



# Exercise 24

Find the first-order necessary conditions and interpret them.

**Answer:**

.

.

We are looking at the RHS of the FONC. The first term is the change in revenue caused by the price change (because of change in quantity supplied). The second term is the change in revenue because the change in the quantity produced. In this case, is the marginal cost. The following is a graphical illustration.

|  |  |
| --- | --- |
|  | The top brownish rectangle shows the loss in revenue because of a price drop , and the pink rectangle shows the gain in revenue because of the increase in the quantity sold at the lower price .  The difference between the two rectangles must be equal to the wage rate . If it is more, the profit maximum has not yet been reached, and if it is less the profit maximizing quantity has been exceeded. |

# Exercise 25

Use the comparative static method to find the slope of the input demand curves. What results do you get?

**Answer:**

From Exercise 24: .

Assume two inputs.

, by Cramer’s rule (with ).

I expect , but I do not know how to obtain this sign from my result. Note that

# Exercise 26

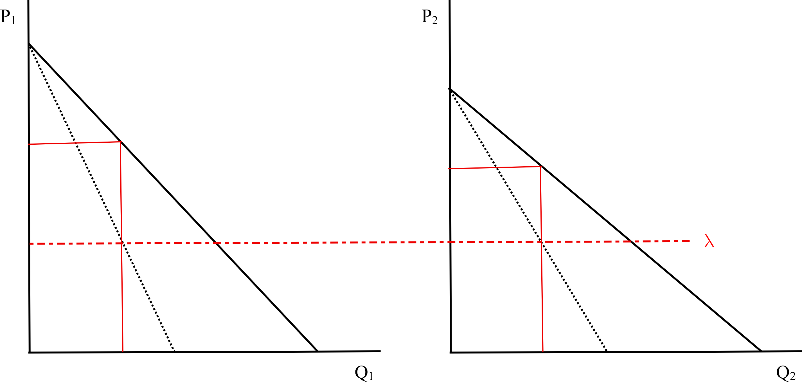
Suppose that a monopolist sells output in two separated markets. The firm faces (inverse) demand curves in the two markets in market 1 and in market 2. The production function is and . Suppose that profit is maximized. Write out the first-order necessary conditions for this problem. What do the first-order conditions tell us? Interpret any multipliers. Draw a graph.

**Answer:**

The first two conditions say that input should be purchased until the input price equals the value of its marginal product.

The second two conditions say that the quantity sold in each market should be such that marginal revenue is the same, namely .

The last condition ensures that the quantity sold in the two markets does not exceed the quantity produced.



# Exercise 27

Suppose that the monopoly acts to maximize revenue (not profit) subject to the condition that profit cannot be negative. What output does the firm decide to produce? Write out the first-order necessary conditions for this problem and draw a graph to illustrate the firm’s choice. For ease of argument, assume revenue is and total cost is .

**Answer:**

; and

If the constraint on profit is met, that is, if , then and we have the familiar condition that maximizes total revenue. If , we have .

The graph is on the next page.

|  |  |
| --- | --- |
|  | The graph suggests that the most likely outcome is . In such a case |

# Exercise 28

In this case (one input and one output), you can draw the combination of and so that profit is constant, the iso-profit line. Suppose the price of output is and the wage rate is ; we assume these are parameters to the producer. . Set profit equal to a constant and then graph the resulting equation (use an axis system with on the horizontal axis and on the vertical axis). What combinations of and yield that constant profit? What happens to iso-profit is rises? If raises?

**Answer:**

|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| If we use more labor, cost goes up and revenue must keep up, which is achieved by selling more and increasing revenue . If rises, the iso-profit line shifts upward, and if rises, the iso-cost curve also shifts upward. | |  |  | | --- | --- | |  |  | | 1 | 1.5 | | 2 | 2.0 | | 3 | 2.5 | | 4 | 3.0 | | 5 | 3.5 | | 6 | 4.0 | | 7 | 4.5 | | 8 | 5.0 | | 9 | 5.5 | | 10 | 6.0 | | 11 | 6.5 | | 12 | 7.0 | | 13 | 7.5 | | 14 | 8.0 | | 15 | 8.5 | |

# Exercise 29

Where would profit maximization be? Find the solution graphically. Write the FONC and interpret it. Your graphical solution should also satisfy the FONC of the profit maximization where . Show how the FONC is illustrated in your graph.

**Answer:**

;

Interpretation: To maximize profit, hire labor services until the marginal value product equal the marginal cost, which in this case is the wage rate.

|  |  |
| --- | --- |
|  | At , the slope of is . is the slope of the labor cost line, . |

# Exercise 30

What are the second-order conditions for this problem? What do they require for the graph? How do the SOSC for this problem relate to the SOSC for the problem of one output and many inputs?

**Answer:**

SOSC: . This means that at the point of maximum profit, the marginal product of labor is positive (FONC) but decreasing (SOSC). The graph of in Exercise 29 shows this property.

The SOSC in this case is the same as in one output and many inputs: In a neighborhood of the maximum, the function must be concave.

# Exercise 31

In this problem, The problem with this expression is that is supposed to depend on , but depends on instead. We can fix this by using the production function, . Invert the production function to write as a function of . Then use this expression to replace in the expression for . Now find the . Combine this expression with the FONC for profit maximization for this problem. What conclusion do you draw?

**Answer:**

We use the Inverse Function Theorem (Hoag, p. 147). Since it follows that in a neighborhood of the optimal solution. Therefore, the Inverse Function’s requirements are met. Therefore, we can write .

; . The FONC is the familiar condition that .

# Exercise 32

Suppose that there is a tax of per unit of output. What impact will there be on the profit maximizing output? What if the tax is a lump sum tax of ? What impact now? Show these taxes in the graph. Include the taxes in your mathematical statement and show that your new FONC agree with the graph.

**Answer:**

; . The FONC is the modified condition that . Compared to no tax, the optimal quantity is smaller and profit is smaller, too.

With a lump sum tax: ; . The FONC is the same as without a tax but profit is reduced by the amount of the lump sum tax.

|  |  |  |
| --- | --- | --- |
|  |  |  |

The slop of is the marginal cost and the slope of is the marginal revenue, which in this case is equal to price (our firm is a price taker).

# Exercise 33

Use the FONC from Exercise 29 and, by comparative statics, find the slope of the input demand. What determines the slope of input demand? Can you use your comparative statics to find the slope of output supply? What do you get?

**Answer:**

;

The slope of input demand is determined by the change in the value of labor’s (input) marginal product. We cannot derive the slope of output supply.

# Exercise 34

What do the indifference curves for Robinson look like? Draw some indifference curves on an axis system with on the horizontal axes and on the vertical axis.

**Answer:**

|  |  |
| --- | --- |
| Graph for Exercise 34 | Graph for Exercise 35 |
|  |  |

# Exercise 35

Write the Lagrange and first-order necessary conditions for the maximum (maximum utility for Robinson Crusoe). Show your first-order conditions in the graph.

**Answer:**

; FONC:

At the optimum, the marginal utility of output must equal the marginal utility of leisure .

For the graph, see above. I assume the production possibilities curve is linear (in red). The indifference curve and the production possibilities curve are tangent to each other.

# Exercise 36

What interpretation do you have for these first-order conditions? How do you interpret the multiplier in this case?

**Answer:**

; FONC: , time at work .

At the optimum, the marginal utility of leisure must be equal to the marginal utility of working:

. The multiplier is the marginal value of more time, .

# Exercise 37

What are the second-order sufficient conditions for this problem?

**Answer:**

SOSC:

The optimum is a maximum.

# Exercise 38

Suppose that there is a tax placed on the output. What impact will this have on the consumption and production of the good? The hours worked? How can you go about addressing this question? Rewrite Lagrange to include the tax. Carry out the comparative statics for this problem to find

**Answer:**

;

. From this follows that at the optimum . This shows that the consumption of leisure will increase and production and consumption of the good will decrease. To show this, denote leisure in the answer without tax by : .

. Thus , and only if

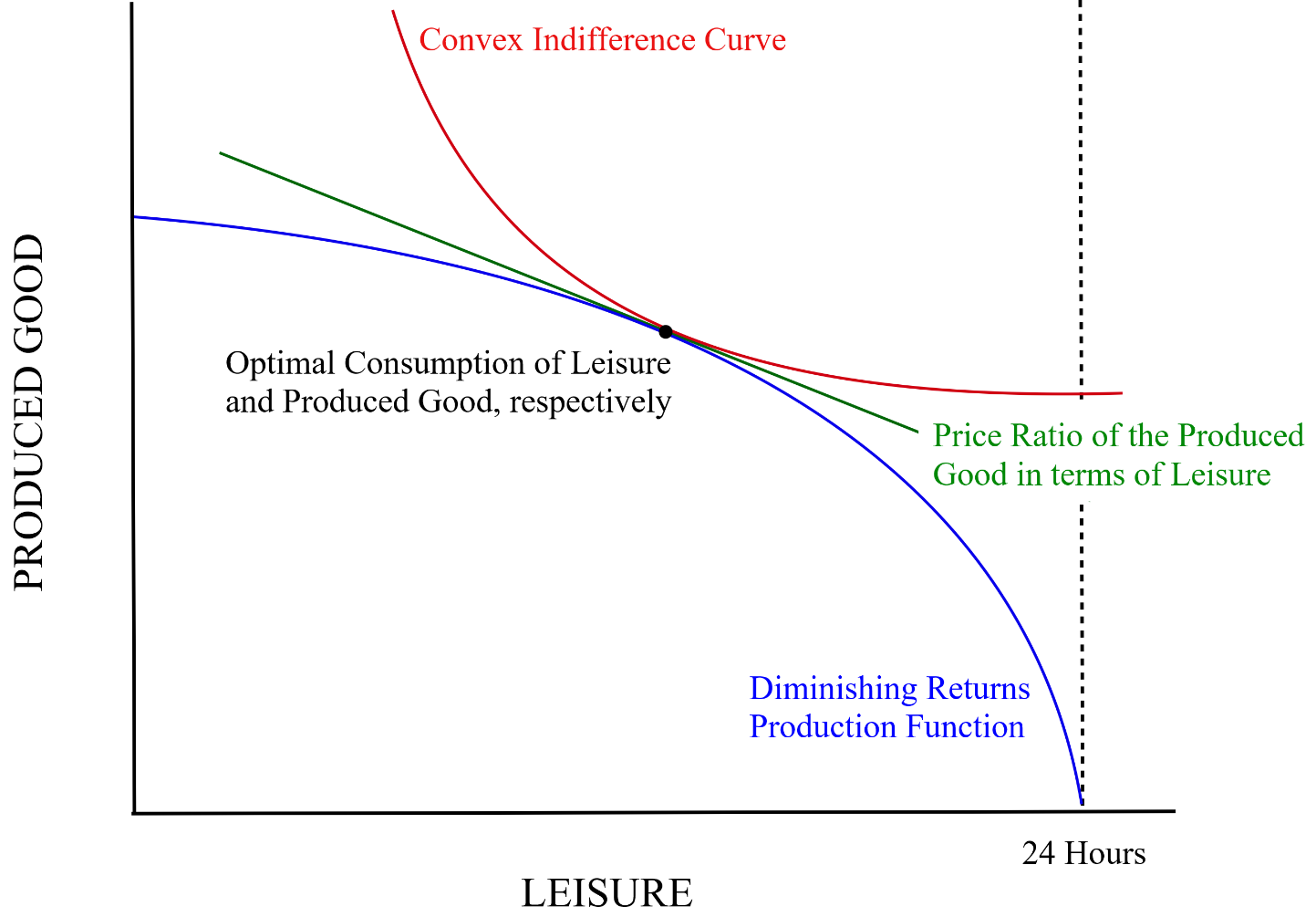
Assume that . Then, using Cramer’s Rule:

Hence, . I expect the sign of this expression to be negative, but I do not know how to show that it is negative.

# Exercise 39

Suppose that Robinson’s production function everywhere satisfies diminishing returns (the first derivative of the production function is everywhere decreasing). What will the production function look like? Suppose also that Robinson’s indifference curves have AsGood sets that are strictly convex. How would you establish that there are prices that exist so that Robinson will choose a combination of output and labor so that both profit and utility are maximum? (HINT: consider a separating hyperplane).

**Answer:**



**Section XI: Game Theory**

# Exercise 1

Compute the best reply functions for each consumer in the following game.

**Answer:**

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  | **2** | | | |
| **1** | **L** | **C** | **R** | **Best Reply** |
| **T** | (0,0) | (1,1) | (1,0) | C |
| **M** | (-1,1) | (0,0) | (-1,-1) | L |
| **B** | (-1,-1) | (-1,0) | (0,1) | R |
| **Best Reply** | T | T | T |  |

Player 1 has a dominant strategy, which is T. In this case, if Player 2 knows that Player has this dominant strategy, the reply that yields the highest return is to play C.

# Exercise 2

Find the best reply functions (and dominant strategies) for this case.

**Answer:**

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  | **2** | | | |
| **1** | **L** | **C** | **R** | **Best Reply** |
| **T** | (1,0) | (1,1) | (1,0) | C |
| **M** | (0,2) | (0,1) | (2,-1) | L |
| **B** | (-1,-1) | (-1,1) | (0,-1) | C |
| **Best Reply** | T | T | M |  |

The highlighted responses do not matter because they will never be chosen (B by Player 1 and R by Player 2 are dominated strategies). Thus, Player 1 will play T and Player 2 will play C.

# Exercise 3

Compute the best reply functions for each consumer in the following game. What do you find for equilibria in each case?

**Answer:**

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  | **2** | | | |
| **1** | **L** | **C** | **R** | **Best Reply** |
| **T** |  |  |  | R |
| **M** |  |  |  | L |
| **B** |  |  |  | L |
| **Best Reply** | B | T | T |  |

Player 1 never plays M and Player 2 never plays C, regardless of the other player’s choice. Thus, M and C are dominated strategies. The two Nash equilibria are B,L and T,R.

# Exercise 4

Compute the best reply functions for both consumers in the above game. Is there a Nash equilibrium?

**Answer:**

|  |  |  |  |
| --- | --- | --- | --- |
|  | **2** | | |
| **1** | **a** | **b** | **Best Reply** |
| **A** |  |  | a |
| **B** |  |  | b |
| **Best Reply** | A | B |  |

There are two Nash equilibria, but it is unclear if they will be chosen without prior communication. This problem is known as the “Battle of the Sexes.” We will later see that there is also a mixed strategy Nash equilibrium, when the players’ choice of strategy has a probability assigned to it.

# Exercise 5

Compute the best reply function in the game shown here. Is there an equilibrium?

**Answer:**

|  |  |  |  |
| --- | --- | --- | --- |
|  | **2** | | |
| **1** | **Hold** | **Rat** | **Best Reply** |
| **Hold** |  |  | Rat |
| **Rat** |  |  | Rat |
| **Best Reply** | Rat | Rat |  |

There is one Nash equilibrium, which is for both players to “Rat.” “Rat” is a dominant strategy. Note that this is not the best possible outcome, which would be obtained if both players “Hold.” This problem is known as the “Prisoners’ Dilemma.”

# Exercise 6

In the Prisoners’ Dilemma, are there dominated strategies?

**Answer:**

“Hold” is a dominated strategy for both players.

# Exercise 7

Finally, consider this game. Are there any dominated strategies? Compute the best reply functions. What outcome will the players have in this case? Where is the equilibrium?

**Answer:**

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  | **2** | | | |
| **1** | **L** | **C** | **R** | **Best Reply** |
| **T** |  |  |  | C |
| **M** |  |  |  | R |
| **B** |  |  |  | L |
| **Best Reply** | M | B | T |  |

There is not pure strategy equilibrium in this case.

# Exercise 8

What is the probability that the outcome is where 1 plays A and 2 plays b? (see page 283)

**Answer:**

The probability is .

# Exercise 9

For the game we just solved, what is the expected value of the game for each player at their Nash mixed strategy equilibrium? Are the expected payoffs different? Find the expected value (for Player 1) of strategy A and of strategy B. Are these the same or different?

**Answer:**

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  |  |  |  |  |
|  |  | **2** | | |
|  | **1** | **a** | **b** | **Best Reply** |
|  | **A** |  |  | b |
|  | **B** |  |  | a |
|  | **Best Reply** | A | B |  |

We know (see derivation in Hoag, pp. 283/4), that the mixed Nash equilibrium is .

1.66667

The expected payoffs are different.

Strategy A:

Strategy B:

The expected payoffs for the two strategies are the same.

# Exercise 10

Find the value of that makes the expected payoff of for Player 2 equal to the payoff from strategy .

**Answer:**

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  |  |  |  |  |
|  |  | **2** | | |
|  | **1** | **a** | **b** | **Best Reply** |
|  | **A** |  |  | b |
|  | **B** |  |  | a |
|  | **Best Reply** | A | B |  |

Strategy : ; Strategy : ;

# Exercise 11

For the game in Exercise 7, find the Nash mixed strategy equilibrium.

**Answer:**

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  | **2** | | | |
| **1** | **L** | **C** | **R** | **Best Reply** |
| **T** |  |  |  | C |
| **M** |  |  |  | R |
| **B** |  |  |  | L |
| **Best Reply** | M | B | T |  |

Strategy T: ; Strategy M: ; Strategy B: .

Strategy L: ; Strategy C: ; Strategy R: .

Both players play each of their strategies with probability .

# Exercise 12

For the game given below, use the process given above to find the mixed strategy Nash equilibrium. What problem do you encounter here?

**Answer:**

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  |  |  |  |  |
|  |  | **2** | | |
|  | **1** | **a** | **b** | **Best Reply** |
|  | **A** |  |  | a |
|  | **B** |  |  | a |
|  | **Best Reply** | B | B |  |

We can already see the problem: Player 1 never plays A, and player 2 never plays b; A and b are dominated strategies. Thus, and .

Strategy A: Strategy B: ; which makes no sense as a value of a probability.

Strategy a: ; Strategy b: , which makes no sense as a value of a probability.

# Exercise 13

For the game below, find the bet reply for each strategy for each player. Are there dominated strategies? Now try to find the mixed strategy Nash equilibrium assuming that each player uses all three strategies. What happens?

**Answer:**

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  | **2** | | | |
| **1** | **L** | **C** | **R** | **Best Reply** |
| **T** |  |  |  | L |
| **M** |  |  |  | R |
| **B** |  |  |  | L |
| **Best Reply** | M | M | B |  |

T is a dominated strategy for player 1 and C is a dominated strategy for player 2

Strategy T: ; Strategy M: ;   
Strategy B: .

Strategy L: ; Strategy C: ; Strategy R: .

We get nonsensical results for the values of the probabilities.

Note: If we eliminate the dominated strategies, we obtain a mixed strategy Nash equilibrium

# Exercise 14

For the following game, find the best reply. Is there a pure strategy Nash equilibrium? Are there any dominated strategies? Find the mixed strategy Nash equilibrium. What do you get? Check the expected values for both players in this case.

**Answer:**

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  |  |  |  |  |
|  |  | **2** | | |
| **1** | |  |  | **Best Reply** |
|  |  |  |  |  |
|  |  |  |  |  |
|  |  |  |  |  |
|  | **Best Reply** |  |  |  |

Player 1 will never use strategy C. It is not a dominated strategy according to Definition XI.3 (p. 277), because . Since C is never used, the probability that it will be used in 0.

Strategy A: ; Strategy B:; Strategy C: .

Strategy a: ; Strategy b:

The mixed strategy is player 1 playing A with probability and B with probability , and player 2 playing a and b with probability , each.

# Exercise 15

Show that for Player 1, if A is played with probability 0.5 and B is played with probability 0.5, the outcome is better than if Player 1 played C.

**Answer:**

If player 1 plays A with probability 0.5 and B with probability 0.5, this player’s expected payoff is . If player 1 plays C, the expected payoff is .

# Exercise 16

Eliminate C and find the mixed strategy Nash equilibrium in the above game.

**Answer:**

Strategy a: Strategy b:

Strategy A: ; Strategy B:

# Exercise 17

What are the best reply functions for this game? Find a mixed strategy Nash equilibrium for this game.

**Answer:**

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|  |  |  |  | | |  |  |
|  |  | **2** | | | | | |
| **1** | | **L** | | **C** | **R** | | **Best Reply** |
|  | **T** |  | |  |  | | L |
|  | **M** |  | |  |  | | L |
|  | **B** |  | |  |  | | R |
|  | **Best Reply** | T | | B | B | |  |

There are two pure strategy Nash equilibria (marked in red).

Each player has a strategy that will not be played in pure strategy Nash equilibrium (marked in grey). Note that M is not dominated, since . Similarly, C is not dominated since .

Strategy T: Strategy M: Strategy B:   
. Use result from line above: . Solve for

Player 2 plays L with probability and R with probability .

Strategy L: ; Strategy C: ;   
Strategy R:   
.  
.

We have a mixed strategy Nash equilibrium with:   
.

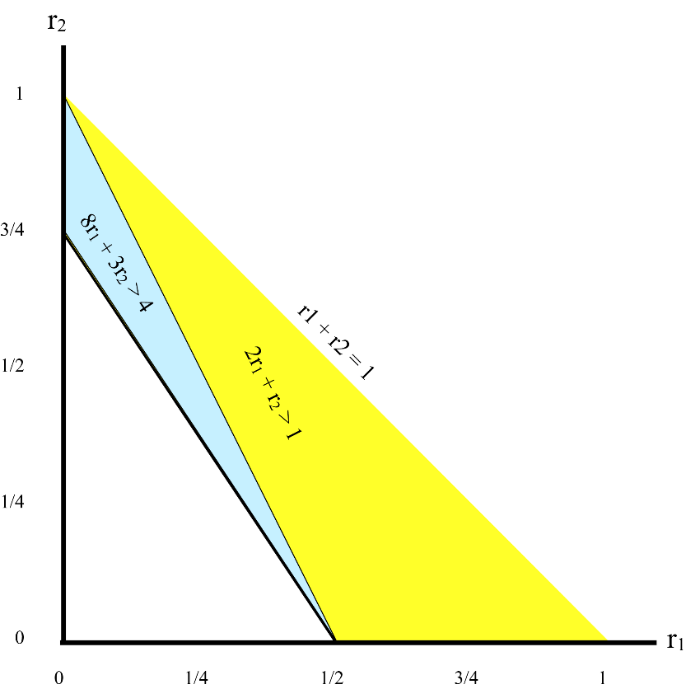
# Exercise 18

Simplify these expressions and graph them on a graph with on the horizontal axis and on the vertical axis. Also, realize that the condition must hold. Where, on the graph, are the value of and that satisfy these conditions?

**Answer:**

The two conditions are and .  
They can be simplified: and .

Picture is on the next page. Note that only the yellow area satisfies both conditions.



# Exercise 19

To continue, B would be dominant if the expected value of B is greater than the expected value of M and greater than the expected value of T. These two conditions can be written as follows:

Simplify these expressions and graph them on a graph with on the horizontal axis and on the vertical axis. Also, realize that the condition must hold. Where, on the graph, are the value of and that satisfy these conditions? Are there values of and so that B is the best choice?

**Answer:**

|  |  |
| --- | --- |
|  | The shaded area shows values of and so that B is the best choice. |

# Exercise 20

M will be dominant if the following conditions hold.

Simplify these expressions and graph them on a graph with on the horizontal axis and on the vertical axis. Also, realize that the condition must hold. Where, on the graph, are the value of and that satisfy these conditions? Now combine the outcome of the last three exercises. Are there conditions where M would be played?

**Answer:**

|  |  |
| --- | --- |
|  | The two conditions have only the point in common which means that . This means that C will not be played. |

# Exercise 21

Complete the analysis showing that C will never be played.

**Answer:**

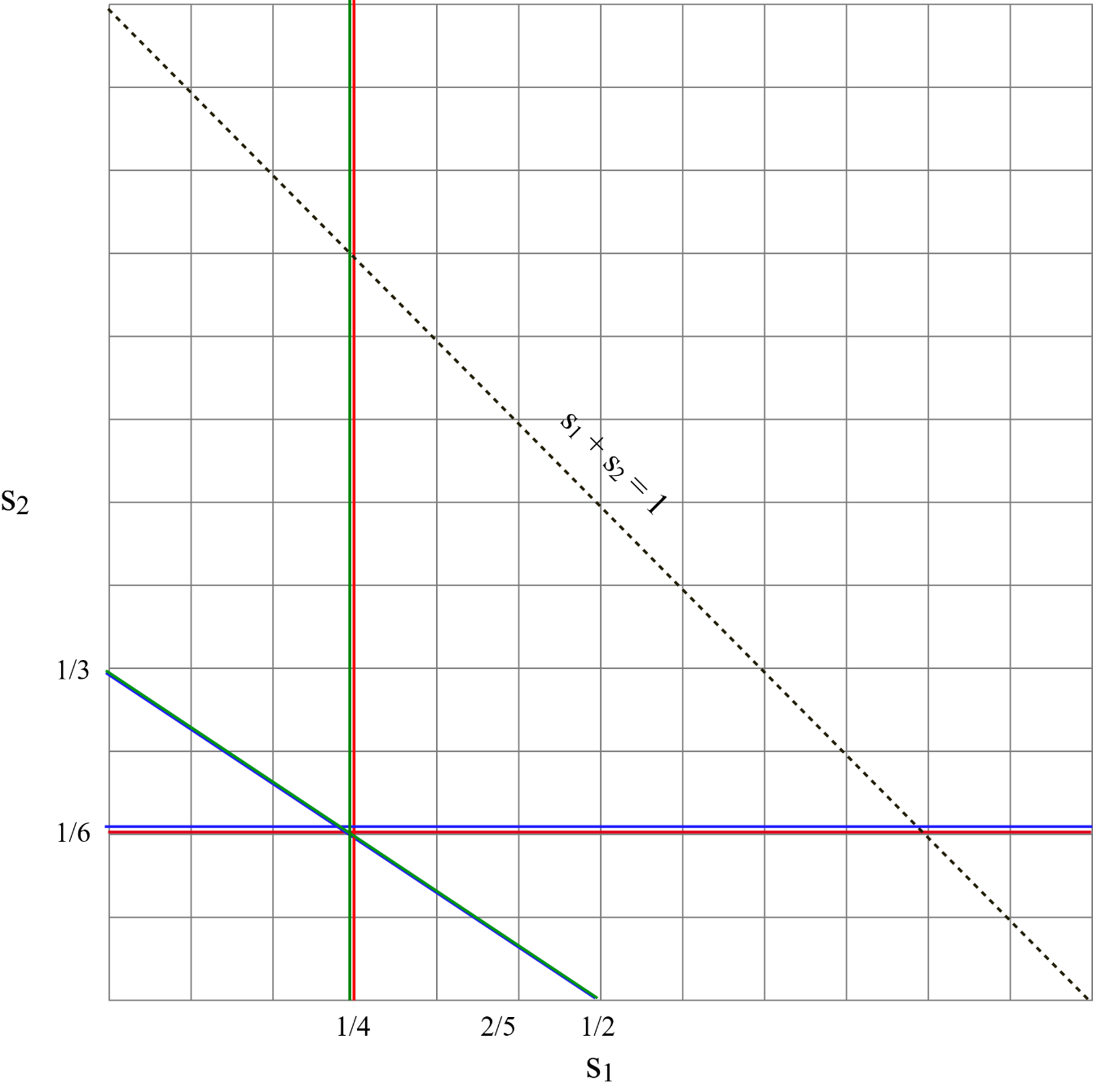
|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|  |  |  |  | | |  |  |
|  |  | **2** | | | | | |
| **1** | | **L** | | **C** | **R** | | **Best Reply** |
|  | **T** |  | |  |  | | L |
|  | **M** |  | |  |  | | L |
|  | **B** |  | |  |  | | R |
|  | **Best Reply** | T | | B | B | |  |

Conditions for L to be better than C and better than R:

Conditions for R to be better than C and better than L:

Conditions for C to be better than L and better than R:

Graph of the borders (lines) of the conditions.



# Exercise 22

Find the best reply for both consumers 1 and 2.

**Answer:**

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|  |  |  |  | | |  |  |
|  |  | **2** | | | | | |
| **1** | | **L** | | **C** | **R** | | **Best Reply** |
|  | **T** |  | |  |  | | L |
|  | **M** |  | |  |  | | C |
|  | **B** |  | |  |  | | L |
|  | **Best Reply** | M | | T | M | |  |

# Exercise 23

Are there any dominated strategies in this case?

**Answer:**

Hoag writes that R is dominated, but I do not think so because . Although player 1 does not chose B, B is not dominated.

# Exercise 24

Are there any Nash equilibria in terms of pure strategies in this game?

**Answer:** No, there are no pure strategies.

# Exercise 25

For this game, find the best responses for each player. Is there a Nash equilibrium in pure strategies? Are there any dominated strategies? Find a mixed strategy Nash equilibrium using all three strategies. What problem do you encounter?

**Answer:**

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|  |  |  |  | | |  |  |
|  |  | **2** | | | | | |
| **1** | | **L** | | **C** | **R** | | **Best Reply** |
|  | **T** |  | |  |  | | C |
|  | **M** |  | |  |  | | L |
|  | **B** |  | |  |  | | R |
|  | **Best Reply** | B | | M | T | |  |

There is no pure strategy Nash equilibrium. There are no dominated strategies.

.  
.

|  |  |  |
| --- | --- | --- |
|  |  | Hence: |

|  |  |  |
| --- | --- | --- |
| ; |  | We cannot find admissible solutions for and . |

See discussion in Hoag: pp. 296-298.

# Exercise 26

Go back to Sec. IV, where finding solutions to linear equations is discussed and based on the discussion there, show that these three equations are linearly dependent.

**Answer:**

|  |  |
| --- | --- |
| Eqn. 1 (from Ex. 25):  Eqn. 2:  Eqn. 3: | We have 3 equations and 2 unknowns. Note that if we subtract the last row from the second row, we get the first row. |

See plot of the 3 equations in Hoag: page 297.

# Exercise 27

More generally, suppose we have three vectors and . And form three new vectors from these as follows.

|  |  |  |
| --- | --- | --- |
|  |  | . |

Show that . This would mean that the three vectors and are linearly dependent.

**Answer:**

# Exercise 28

Show that in this game, and . What is the expected value of T? B? C? R? Is ? ? Do the two players have the same expected value for the game? Should they?

**Answer:**

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  |  |  |  |  |
|  |  | **2** | | |
|  | **1** | **C** | **R** | **Best Reply** |
|  | **T** |  |  | C |
|  | **B** |  |  | R |
|  | **Best Reply** | B | T |  |

if .

if

only for and only for ,

Expected Value of Game, Player 1:

Expected Value of Game, Player 2:

Thus, the two players do not have the same expected value of the game. This does not surprise because each outcome has a positive payoff for Player 2 and except for outcome , Player 2’s payoff is larger than Player 1’s payoff for every combination of strategies.

# Exercise 29

Case 4. In the case below we have Player 1 using only T, M, and B, while Player 2 uses L and C. We have the following game. Is there a pure strategy equilibrium? Are there dominant strategies? Look for the mixed strategy. Are the excluded strategies preferred?

**Answer:**

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  |  |  |  | Best Response |
|  |  | **2** | | |
|  | **1** | **L** | **C** |  |
|  | **T** |  |  | C |
|  | **M** |  |  | L |
|  | **B** |  |  | C |
| Best Response |  | B | M |  |

T is a dominated strategy as and .

There is no pure strategy equilibrium.

. . .  
.  
, which is an inadmissible solution.  
.

.  
 and , which is an inadmissible result.

To proceed, we eliminate strategy T:

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  |  |  |  | Best Response |
|  |  | **2** | | |
|  | **1** | **L** | **C** |  |
|  | **M** |  |  | L |
|  | **B** |  |  | C |
| Best Response |  | B | M |  |

From above: . .  
.

.

Mixed strategy: Player 1 plays M and B with probability 2/3 and 1/3, respectively. Player 2 plays L and C with probability 1/5 and 4/5, respectively.

Value of game, player 1:

Value of game, player 2:

# Exercise 30

Case 5. In the case below we have Player 1 randomizing over T, M, and B, while Player 2 utilizes L and R. We have the following game. Is there a pure strategy equilibrium? Are there dominant strategies? Look for the mixed strategy. Are the excluded strategies preferred?

**Answer:**

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  |  |  |  | Best Response |
|  |  | **2** | | |
|  | **1** | **L** | **R** |  |
|  | **T** |  |  | L |
|  | **M** |  |  | L |
|  | **B** |  |  | R |
| Best Response |  | B | T |  |

M is a dominated strategy as and . In addition, and

There is no pure strategy Nash equilibrium.

. . .  
, which is an inadmissible solution.  
.  
.

.  
 and , which is an inadmissible result.

To proceed, we eliminate strategy M:

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  |  |  |  | Best Response |
|  |  | **2** | | |
|  | **1** | **L** | **R** |  |
|  | **T** |  |  | L |
|  | **B** |  |  | C |
| Best Response |  | B | M |  |

From above: . . .

. .

Mixed strategy: Player 1 plays T and B with probability 3/5 and 2/5, respectively. Player 2 plays L and R with probability 1/3 and 2/3, respectively.

Value of game, player 1:

Value of game, player 2:

For player 1 the value of this game is the same in Case 4. For player, the value of this game is higher.

# Exercise 31

Case 6. In the case below we have Player 1 randomizing over T, M, and B, while Player 2 randomizes over C and R. We have the following game. Is there a pure strategy equilibrium? Are there dominant strategies? Look for the mixed strategy. Are the excluded strategies preferred?

**Answer:**

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  |  |  |  | Best Response |
|  |  | **2** | | |
|  | **1** | **C** | **R** |  |
|  | **T** |  |  | C |
|  | **M** |  |  | C |
|  | **B** |  |  | R |
| Best Response |  | M | T |  |

B is not used but it is not a dominated strategy.

There is one pure strategy Nash equilibrium, namely (M, C).

. . .  
.  
.  
.

.  
 and , which is an inadmissible result.

To proceed, we eliminate strategy B:

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  |  |  |  | Best Response |
|  |  | **2** | | |
|  | **1** | **C** | **R** |  |
|  | **T** |  |  | C |
|  | **M** |  |  | C |
| Best Response |  | M | T |  |

R is a dominated strategy. There also exists one pure strategy Nash equilibrium: (M, C).

. . .

. , which is an inadmissible result.

There is no mixed strategy equilibrium. Since player 2 always plays C, regardless of what player 1 does, player 1’s best response is M. Hence, the pure strategy Nash equilibrium: (M, C) is the only Nash equilibrium.

Value of game for player 1 is 2 and the value of the game for player 2 is 1.

For player 1, the value of this game is higher than that of Case 4 and Case 5.

For player 2, the value of this game is lower than that of Case 4 and Case 5.

# Exercise 32

Show that M, B with C, R is a Nash equilibrium.

Answer:

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  |  |  |  | Best Response |
|  |  | **2** | | |
|  | **1** | **C** | **R** |  |
|  | **M** |  |  | C |
|  | **B** |  |  | R |
| Best Response |  | M | B |  |

is a pure strategy Nash equilibrium

.

Expected value of game if mixed strategy is use:

Expected value of game for player 1:

Expected value of game for player 2:

# Exercise 33

Show that the other six solutions are not Nash equilibria.

**Answer:** I do not understand the question.

# Exercise 34

Suppose we have a duopoly where two firms produce similar goods, but the goods are not exactly the same good. Further suppose that they have each hired an advertising firm. Each advertising firm has told its client that they propose three possible advertising campaigns, a low cost, medium cost, and high cost campaign. Of course, the firms can still choose to not advertise (Zero). Neither firm knows what the other will do, but they each have some idea of what the profits will be for themselves and for their rival. The information is provided below where the pairs of numbers of profits to Firm 1 and to Firm 2. Are there any dominant strategies? Is there a Nash equilibrium in pure strategies? Find the Nash equilibrium for this game.

**Answer:**

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  | **Firm 2** | | | | |
| **Firm 1** | **Zero – r1** | **Low – r2** | **Medium – r3** | **High – 1-r1-r2-r3** | **Best Reply** |
| **Zero** |  |  |  |  | High |
| **Low** |  |  |  |  | High |
| **Medium** |  |  |  |  | Medium |
| **High** |  |  |  |  | Low |
| **Best Reply** | High | Medium | High | Medium |  |

There is no pure strategy Nash equilibrium.   
Dominated strategies: Player 1 – Medium is always better than Zero or Low  
Dominated strategies: Player 2 –Low is always better than Zero

If player 1 eliminates strategies Zero and Low and player 2 eliminates strategy Zero, we have the following game:

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  | **Firm 2** | | | |
| **Firm 1** | **Low – r2** | **Medium – r3** | **High – 1-r2-r3** | **Best Reply** |
| **Medium** |  |  |  | Medium |
| **High** |  |  |  | Low |
| **Best Reply** | Medium | High | Medium |  |

In this modified game, High is a dominated strategy for player 2. If we eliminate this strategy, we have the following game:

|  |  |  |  |
| --- | --- | --- | --- |
|  | **Firm 2** | | |
| **Firm 1** | **Low** **r** | **Medium** **1 – r** | **Best Reply** |
| **Medium s** |  |  | Medium |
| **High 1 – s** |  |  | Low |
| **Best Reply** | Medium | High |  |

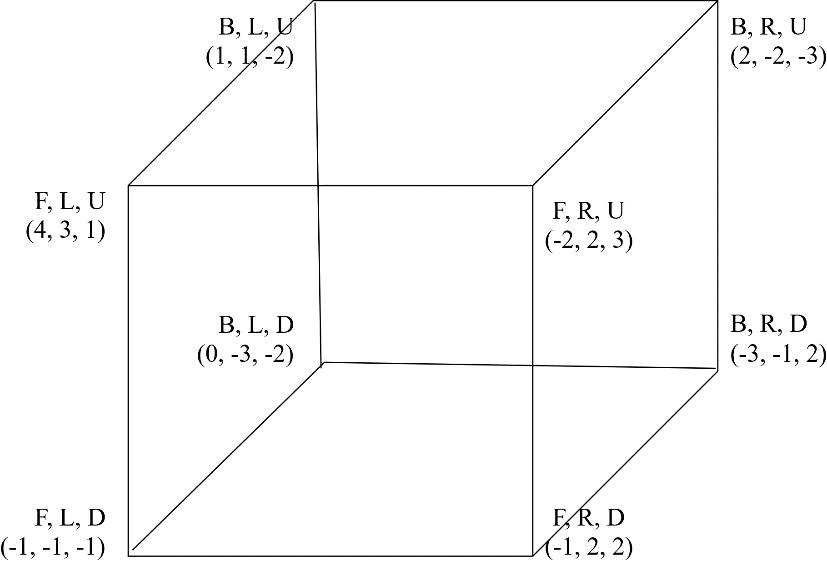
Player 1

Player 2

The probabilities and describe the mixed strategy Nash equilibrium.

# Exercise 35

Here is a game with a unique pure strategy Nash equilibrium. Where is it? Explain how you know.



**Answer:**

BLU: Player 1 is better off at FLU  
BRU: Player 2 is better off at BLU and Player 3 at BRD  
BLD: Player 2 is better off at BRD  
BRD: Player 1 is better off at FRD  
FLD: Player 1 is better off at BLD, Player 2 at FRD, and Player 3 at FLU  
FRD: Player 3 is better off at FRU  
FRU: Player 1 is better off at BRU and Player 2 at FLU  
FLU: No player can make a unilateral move that makes her better off. FLU is the pure strategy Nash equilibrium.,

# Exercise 36

In the above example, for (Hoag, p. 308) the case where Player 1 plays only B, show that and are the solutions. Also show that the expected value of the excluded alternative, F, is greater than the expected value of B.

**Answer:**

|  |  |
| --- | --- |
|  | If Player 1 only plays B, we have the game: |

Let and .

.

.

.  
.  
This shows that .

# Exercise 37

In the above example, when Player 2 plays only R, show that .

**Answer:**

|  |  |
| --- | --- |
|  | Reminder: .  . . . |

# Exercise 38

In the above example, show that when Player 2 plays L, we obtain and . Further, show that the expected value of the excluded alternative, R, is less than the expected value of L.

**Answer:**

|  |  |
| --- | --- |
|  | Reminder: . .    . |

# Exercise 39

It should be clear to you by now that if there is more than one pure strategy equilibrium, then there is almost surely a mixed strategy equilibrium. In the game below, you know that there are two pure strategies, FLU and BRD. But there is also a mixed strategy equilibrium. Find it.

**Answer:**

|  |  |
| --- | --- |
|  | . |
| Let Player 1 play F:  Let Player 1 play B: | Let Player 1 play F:  Let Player 1 play B: |
| Let Player 2 play L:  Let Player 2 play R: | Let Player 2 play L:  Let Player 2 play R: ; |
| Let Player 3 play U: ;  Let Player 3 play D: | Let Player 3 play U: ;  Let Player 3 play D: |

Mixed strategy Nash equilibrium: Player 1 – F R ; Player 2 – L ; R ; Player 3 – U always.

# Exercise 40

But even of more interest is the case of a case of a single pure Nash equilibrium, and still there may be a mixed strategy equilibrium. Here there are two mixed strategy equilibria. Find them!

Answer:

|  |  |
| --- | --- |
|  | . |
| Let Player 1 play F:  Let Player 1 play B: | Let Player 1 play F:  Let Player 1 play B: |
| Let Player 2 play L:  Let Player 2 play R: | Let Player 2 play L:  Let Player 2 play R: |
| Let Player 3 play U:  Let Player 3 play D: | Let Player 3 play U:  Let Player 3 play D: |

The mixed strategy Nash equilibrium: Player 1 plays F with probability and B with probability . Player 2 always plays R. Player 3 plays U with probability and D with probability .

# Exercise 41

Suppose that we have a monopoly with a linear demand, , where is output. Assume that the firm has constant marginal cost and zero fixed cost, so total cost . Find the profit maximizing output and price that will prevail in the market. Note that to do the problem in terms of output who will have to use demand to find the average revenue.

**Answer:**

.

.

# Exercise 42

Write out the FONC for this (duopoly) profit maximization problem. Note that the condition depends on the value of . Solve the first-order condition for .

**Answer:**

. .

.

# Exercise 43

Write out profit for Firm 2 and find the best response function for Firm 2. Assume that both firms face the same marginal cost, .

**Answer:**

. ; ;

.

# Exercise 44

Solve the best response functions found in Exercise 42 and 43 to find the optimal outputs for each firm.

|  |  |
| --- | --- |
| **Answer:**  ;  .  . |  |

# Exercise 45

What difference would occur of the two firms have different costs? Suppose Firm 1’s total cost is and Firm 2’s total cost is . Do the profit maximization for both firms, find the best response function s, and solve the best response functions for the optimal outputs. How is the outcome different from the case when the costs are the same?

**Answer:**

We use the results from Exercises 43 and 44 and replace the expression for cost.

; .

Optimal output for Firm 1:

.

.

Optimal output for Firm 2:

.

if , and vice versa. If the costs are the same, the optimal outputs are the same.

# Remark, p. 317

We should observe that the solution to the Cournot duopoly is a Nash equilibrium. How can we tell?

**Answer:**

Each player’s output generates the largest profit given the other player’s output decision.

# Exercise 46

Graph the best response functions for the two firms. Where do they intersect? (HINT: Put on the horizontal axis and on the vertical axis and graph the two equations.)

|  |  |
| --- | --- |
| **Answer:**  The best response functions intersect at the Nash equilibrium. |  |

# Exercise 47

Find the first-order necessary conditions for this (see below) problem.

.

**Answer:**

.

# Exercise 48

Find the best response function for Firm 1.

**Answer:**

.

# Exercise 49

Write out the profit function for Firm 2; assume quadratic costs. Find the first-order necessary conditions and the best response function for Firm 2.

**Answer:**

Profit function: .

FONC: .

Best response function:

# Exercise 50

What is the Nash equilibrium for this problem? How is this outcome different from the previous case?

**Answer:**

;

;

.

.

if . The best response of Firm 1 (2) is still linear with respect to .

# Exercise 51

Write out the first-order necessary conditions and the best response function for this problem.

**Answer:**

# Exercise 52

Carry out the same computation for the case where costs are low.

**Answer:**

# Exercise 53

Why would a firm not be producing the same good and choosing price? What is arbitrage and what role does it play here?

**Answer:**

Goods manufactured by different firm are rarely identical, even if no effort at product differentiation is being made. The causes for “accidental” differences include differences in production processes and technologies. Of course, many difference are deliberate, to target a specific market segment. Since tastes are not uniform, producer have opportunities to gain limited market power over the targeted segment. This is particularly evident in the case of fashion, automobiles, housing, and in real estate, more generally.

When a company has market power, is no longer equal , and the company maximizes profit by choosing the price such that .

Arbitrage is the exploitation of price differences between markets by buying from the low-price market and selling in the high-price market. In this case, though goods are not identical, they are substitutes (think a Toyota vs. a Honda automobile). This limits the size of the price spread between different producers.

# Exercise 54

Write out the profit function for each firm. Find the FONC for a profit maximum for each firm and solve for the reaction (best reply) functions. What are the equilibrium prices? Observe that this is a Nash equilibrium. Assume constant marginal cost, and , respectively, and zero total fixed cost.

**Answer:**

# Exercise 55

Re-do the Bertrand problem in terms of quantities. Write the profit functions in terms of and . What do you need to do first?

**Answer:**

# Exercise 56

Start with the Cournot model (remark before Exercise 42). Now replace in Firm 1’s profit function with Firm 2’s reaction function and find the profit maximizing choice for Firm 1. Assume that Firm 1 produces that quantity. What output will Firm 2 produce? What will be the price? How is this outcome different from the Cournot duopoly?

**Answer:**

. We found Firm 2’ best response function to be (Exercise 43). Therefore:

.

Firm 1’s Output: .

Firm 2’s Output: . Firm 2 produces one half of what Firm 1 produces.

Total Output: .

Price: .

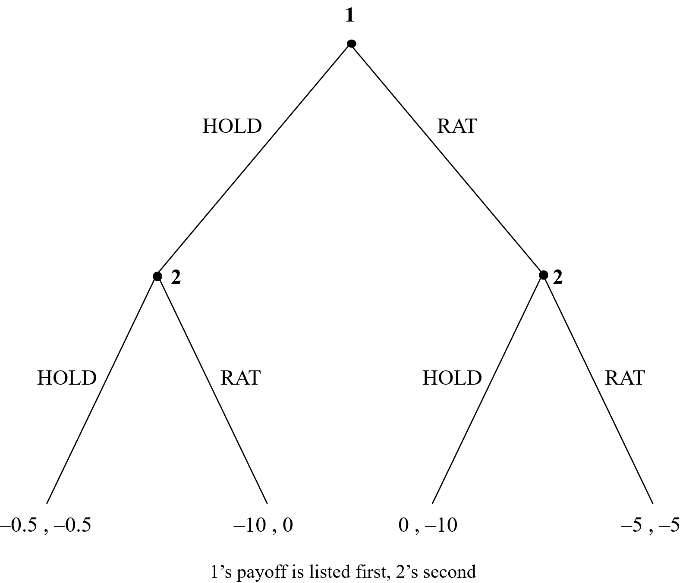
Comparison with Cournot Duopoly: . Both firms produce the same amount. The Cournot Duopoly results in a smaller total production, :

.

# Exercise 57

Draw the Prisoner’s Dilemma in extensive form.

**Answer:**



**Section XIII: Applications**

# Exercise 1

Suppose that the firm does not have a fixed endpoint, that is, that is really infinity. How does this fact change the first-order conditions?

**Answer:**

We no longer need condition (3.c).

# Exercise 2

What if technology does not change so that the production function does not change over time. Does this in any way change the outcome?

**Answer:**

It does not change the mathematics of the problem and first-order necessary conditions. The reason is that the marginal product of capital is not constant because the amount of capital changes. Therefore, the time subscripts in cannot be dropped, both are necessary.

# Exercise 3

Suppose that the firm only exists for today and tomorrow. First, assume that the price of output tomorrow will either be high or low , but the firm does not know which. The firm knows that the probability of is and the probability of is . How would you formulate the firm’s choice problem now?

**Answer:**

S.T.

# Exercise 4

We have assumed that the firm is competitive in the purchase of inputs. What if the firm faces an upward sloping supply curve for investment? How does this change the analysis?

**Answer:**

In such a case we have . We substitute this expression into the Lagrangean function. Then,

# Exercise 5

Rewrite the first-order conditions in term of marginal revenue and marginal product.

**Answer:**

# Exercise 6

Use the envelope Theorem to interpret .

**Answer:**

The terms in red are all zero because they FONC. The last term equals . We assumed that the constraint is binding, so that . If the constraint is not binding, then it has no influence on the profit function and . In either case, we can interpret as the shadow price of the pollution allowance. It tells us by how much profit changes if we marginally change the pollution allowance, . In other words, .

# Exercise 7

Multiply out these matrices (on p. 354) to show that we get the matrices above (on top of p. 354).

**Answer:**

# Exercise 8

We know that (how do we know this?), but we cannot determine the sign of . Write out that determinant and see why we cannot know.

**Answer:**

We know that because the matrix has full rank.

Unless we know the magnitudes of the two terms, we cannot determine the sign of their sum.

# Exercise 9

Use the Envelope Theorem to find how profit changes when changes.

**Answer:**

We solved the problem in Exercise 6, when we showed that.

# Exercise 10

In this problem (pp. 355-358) we have assumed that the advertising affects both demand curves. How would the analysis change if only affected the first firm’s demand and only affected the second firm’s demand?

**Answer:**

In this case:

and

In summary, the mathematics of the problem hardly change (the subscripts of several parameters we used have changed, however).

# Exercise 11

What if the demand curves are not assumed to be linear, but are more general? What difference would this make in the computations?

**Answer:**

Without additional assumptions about the demand curves, it will be impossible to get explicit comparative-static results. The more general the function, the fewer specific results can be derived.

Assume that we can express both prices as functions of quantities produced and advertising efforts:

and . Then,

and .

In this general form, it is not clear if we can solve the system for and .

# Exercise 12

Show that (in the problem on pp. 358-361) is not equal to if is not equal to

and .

**Answer:**

Assume and . Equation is satisfied, but Equation equals by assumption. Therefore, if , must hold or the FONC are not satisfied.

Assume . Equation is satisfied if . Equation is satisfied if . Solve for : . This shows that if then , and if then

1. On page 231 [↑](#footnote-ref-1)